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On Various Age and Residual Life Distributions Associated with the M/G/1 Queue

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Abstract: We show how a simple modification of the time-dependent Little's law can be used to study various types of joint age and residual life distributions associated with any customer waiting in line in equilibrium in a work-conserving M/G/1 queue operating under the first-come-first-served service discipline, not necessarily the customer receiving service. This addresses an open question posed at the end of (Adan and Haviv, Stochastic Models, 2009). We also analyze the joint distribution of the number of customers in the system at time t, the remaining amount of work possessed by the customer currently in service at time t, the amount of work that has already been processed by the customer currently in service at time t, and the amount of time the current customer in service at time t spent waiting in the queue.

Keywords: Age distribution, M/G/1 queue, Residual life distribution.

1. Introduction

Suppose a customer arrives to a stable M/G/1 queueing system while that system is in equilibrium, and she observes n customers already present in the system upon arrival. Given this information, what is the distribution of the remaining amount of time the customer currently in service will spend in service? This is a useful distribution to understand, because the arrival can use that information to determine her waiting time distribution (and likewise, her sojourn time distribution) in the system. Early studies on properties of this distribution, as well as analogous distributions for related models include Wishart [25], Mandelbaum and Yechiali [21], Asmussen [5], Fakinos [13], Boxma [10], with both Sigman and Yechiali [24] and Adan and Haviv [4] being examples of more recent studies on aspects of this distribution.

Interest in such questions/distributions grew around 2008, starting with the work of Kerner [17], where properties of the conditional distribution of the remaining service time of the customer in service, given an arrival observed n customers in the system upon arrival were established for the case where customers are instead assumed to arrive in accordance to a *state-dependent* Poisson process, meaning the arrival rate is a function of the number of customers present in the system. This generalization is important for many reasons, one

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major reason being that the state-dependent arrival model can represent situations where customers may behave more strategically, by choosing not to join the system with some probability depending on the number of customers encountered upon arrival. This question of obtaining good joining strategies was investigated further in Kerner [18]. Other work that address joint distributions of queue-length and residual services, as well as other types of joint distributions, for both the M$_n$/G/1 queue and various generalizations include Abouee-Mehrizi and Baron [1], Oz et al. [22, 23], and Economou and Manou [12].

Our research in this area was inspired by a question posed at the end of Adan and Haviv [4]: if, within the context of the M/G/1 queue, a customer arrives to the system and observes n customers in equilibrium, what is the distribution of the amount of time the kth customer in line has spent waiting in the queue, for $k \in \{2, 3, \ldots, n\}$? We will show that this distribution can be obtained by modifying an idea that has been used to derive a time-dependent form of the distributional Little's law, which dates back to the work of Bertsimas and Mourtzinou [8] and was reexamined through the use of Palm distributions in [16]. Before we study this conditional joint distribution, we will first explain how the distributional Little's law can be used to derive a joint double transform of the number of customers present in the system at time t, and three other random variables, namely the amount of time the customer currently receiving service at time t (i) waited in the queue; (ii) spent in service before time t; and (iii) how much longer past time t that customer will spend receiving service. This double transform, while reasonably tractable, admits a complicated form, so later we will derive a few more detailed results for the case where the queueing system has reached equilibrium, where the last result we derive will address the question from [4] mentioned above.

2. Preliminaries

We begin by briefly describing the M/G/1 queue, as well as the notation we will use to model various aspects of this queueing system. Suppose customers arrive to a single-server queueing system in accordance to a homogeneous Poisson process $\{N(t); t \geq 0\}$ having rate λ and points $\{T_n\}_{n\geq 1}$, so that for each integer $n \geq 1$, T_n represents the arrival time of the nth arrival to the system. The connection between the counting process $\{N(t); t \geq 0\}$ and its points $\{T_n\}_{n\geq 1}$ is clear from the following observations: for each real number $t \geq 0$,

$$N(t) := \sum_{n=1}^{\infty} \mathbf{1}_{\{T_n \leq t\}}$$

and likewise, for each integer $n \geq 1$,

$$T_n := \inf\{t \geq 0 : N(t) \geq n\}.$$

Furthermore, we define, for each Borel set B,

$$N(B) := \sum_{n=1}^{\infty} \mathbf{1}_{\{T_n \in B\}} = \int_B N(ds)$$

so that $N(B)$ denotes the number of arrivals that occur within B, and $N(ds)$ represents Lebesgue-Stieltjes integration, which is well-defined on a path-by-path basis due to the non-decreasing paths of $\{N(t); t \geq 0\}$. We also slightly abuse notation by defining $N(s,t] := N((s,t])$ in an attempt to make many of the mathematical expressions we will encounter in our analysis more readable.

Next, assume that for each integer $n \geq 1$, the nth arrival brings a generally distributed amount of work B_n for processing. The sequence of random variables $\{B_n\}_{n\geq 1}$ is assumed to be both i.i.d. and independent of the arrival process, and for each $n \geq 1$, B_n has a CDF F satisfying $F(0) = 0$. We also let $\beta : \mathbb{C}_+ \to \mathbb{C}$ denote the Laplace-Stieltjes Transform (LST) of F, which is defined as

$$\beta(\alpha) := \mathbb{E}[e^{-\alpha B_1}] = \int_{[0,\infty)} e^{-\alpha x} F(dx)$$

where $F(dx)$ again represents Lebesgue-Stieltjes integration with respect to the probability distribution associated with F, and $\mathbb{C}_+ := \{\alpha \in \mathbb{C} : Re(\alpha) > 0\}$ represents the open halfspace of complex real numbers having positive real part.

Finally, we assume throughout that the server processes work in a First-Come-First-Served (FCFS) manner. It will help to think of the customers present in the system (either waiting or currently being served) as occupying various 'slots' in the system, where the customer in slot 1 is the customer being served, the customer in slot 2 is the next to be served, and so on. Whenever the customer in slot 1 completes his/her service, the customer formerly in slot k moves to slot $k-1$, for each integer $k \geq 2$, and if a new arrival to the system encounters k customers in the system upon arrival, he/she will occupy slot $k+1$ upon joining the queue.

For each real number $t \geq 0$, let $Q(t)$ denote the number of customers currently present in the system (waiting or in service) at time t, and let $W(t)$ denote the total amount of unfinished work present in the system at time t. Next, let $R_{1,s}(t)$ denote the remaining amount of work possessed by the customer receiving service at time t, let $A_{1,s}(t)$ denote the amount of work of the customer receiving service at time t that has been processed by the server, and let $A_{1,q}(t)$ denote the amount of time the customer currently experiencing service at time t waited in the queue before he/she began receiving service. When there is no customer present in the system at time t, we set $A_{1,s}(t) = A_{1,q}(t) = R_{1,s}(t) = 0$. More generally, for each integer $k \geq 2$, we let $A_{k,q}(t)$ denote the amount of time the customer present in slot k at time t has spent in the queue, and we let $R_{k,q}(t)$ denote the remaining amount of time the customer present in slot k will spend *waiting* (not including the time spent in service!) in the queue. Again, $A_{k,q}(t) = R_{k,q}(t) = 0$ when there is no customer present in slot k at time t.

It may be the case that there are customers present at time zero: if there are $Q(0) = n_0$ such customers, we will assume that the customer present in slot k at time zero possesses an amount of work $B_k^{(0)}$. The random variables $\{B_k^{(0)}\}$ are assumed to also be i.i.d. with CDF F, and independent of all other random elements. Recall also that the traffic intensity of the system is $\rho := \lambda \mathbb{E}[B_1]$: it is well-known that the M/G/1 queue is stable if and only if $\rho < 1$.

The workload process $\{W(t); t \geq 0\}$ of the M/G/1 queue plays a major role in the development of our main results, which is convenient because the workload process of the M/G/1 is by now well-understood, see e.g. Abate and Whitt [3]. Indeed, computable expressions can be derived for both the Laplace transform $\phi_{n_0;0} : \mathbb{C}_+ \to \mathbb{C}$ of the emptyness probability, as well as the double transform $\varphi_{n_0} : \mathbb{C}_+ \times \mathbb{C}_+ \to \mathbb{C}$ of the workload. These quantities are formally defined as

$$\varphi_{n_0;0}(\alpha) := \int_0^\infty e^{-\alpha t}\mathbb{P}_{n_0}(Q(t) = 0)dt, \quad \phi_{n_0;W}(\alpha, \gamma) := \int_0^\infty e^{-\alpha t}\mathbb{E}_{n_0}[e^{-\gamma W(t)}]dt, \quad \alpha \in \mathbb{C}_+$$

and computable expressions for these quantities are given in Theorem 2.1 given below. Both of these transforms can be expressed in terms of both the LST β as well as the LST $\pi : \mathbb{C}_+ \to \mathbb{C}$ of the busy period $\tau_0 := \inf\{t \geq 0 : Q(t) = 0\}$ when $Q(0) = 1$ and that single customer present at time zero possesses a random amount of work having CDF F. In other words,

$$\pi(\alpha) := \mathbb{E}_1[e^{-\alpha\tau_0}], \quad \alpha \in \mathbb{C}_+.$$

Readers should recall that π satisfies the fixed-point equation

$$\pi(\alpha) = \beta(\alpha + \lambda(1 - \pi(\alpha))) \tag{1}$$

which can be used to develop an iterative procedure for evaluating $\pi(\alpha)$, for each $\alpha \in \mathbb{C}_+$, see Abate and Whitt [2], as well as [15] which establishes the procedure with a coupling argument.

Theorem 2.1. *The transforms $\varphi_{n_0;0}$ and $\phi_{n_0;W}$ are as follows: for each $\alpha, \gamma \in \mathbb{C}_+$,*

$$\varphi_{n_0;0}(\alpha) = \frac{\pi(\alpha)^{n_0}}{\kappa(\alpha)} \tag{2}$$

and

$$\phi_{n_0;W}(\alpha, \gamma) = \frac{\pi(\alpha)^{n_0}}{\kappa(\alpha)}\left[\frac{\kappa(\alpha) - \gamma}{\alpha - \psi(\gamma)}\right] + \frac{\beta(\gamma)^{n_0} - \pi(\alpha)^{n_0}}{\alpha - \psi(\gamma)} \tag{3}$$

where the functions $\psi : \mathbb{C}_+ \to \mathbb{C}$, $\kappa : \mathbb{C}_+ \to \mathbb{C}$ are defined as

$$\kappa(\alpha) := \alpha + \lambda(1 - \pi(\alpha)), \quad \psi(\gamma) := \gamma - \lambda(1 - \beta(\gamma))$$

for each $\alpha, \gamma \in \mathbb{C}_+$.

These expressions for $\phi_{n_0;W}(\alpha)$ and $\varphi_{n_0;0}(\alpha, \gamma)$ can be derived quickly using recent results from [15], but they are equivalent to previously derived expressions for these transforms, see e.g. Lucantoni et al. [20], where the results derived there hold in a more general context.

It is also well-known that when $\rho < 1$, the distribution of $W(t)$ converges in distribution, as $t \to \infty$, to the law of a random variable $W(\infty)$ we refer to as the steady-state workload. The following corollary shows that the LST of $W(\infty)$ can be expressed in terms of the arrival rate λ and the LST β of the service time distribution.

Corollary 2.1. *(Pollaczek-Khintchine Formula) The steady-state workload $W(\infty)$ has LST*

$$\phi_{W(\infty)}(\gamma) := \mathbb{E}[e^{-\gamma W(\infty)}] = \frac{(1-\rho)\gamma}{\gamma - \lambda(1-\beta(\gamma))}. \tag{4}$$

Corollary 2.1 is even more well-known than Theorem 2.1, but it can be derived from Theorem 2.1 by multiplying $\phi_{n_0;W}(\alpha,\gamma)$ by α, then letting α approach zero from above. An interesting probabilistic interpretation of the distribution of $W(\infty)$ can be found in Cooper and Niu [11], where the authors study the workload process under the assumption where the server processes work under the Last-Come-First-Served Preemptive-Resume discipline: placing this extra assumption on the system does not affect the behavior of the workload process, as it behaves the same for all work-conserving service disciplines.

We conclude this section by recalling the following well-known result for the M/G/1 queue.

Proposition 2.1. *Let $Q_q(\infty)$ denote the steady-state distribution of the number of customers waiting in the system (i.e. not including the customer in service). Then for each z satisfying $Re(z) < 1$,*

$$\mathbb{E}[z^{Q_q(\infty)}] = \mathbb{E}[e^{-\lambda(1-z)W(\infty)}]. \tag{5}$$

Proposition 2.1 follows from the Distributional Little's law discussed in Bertsimas and Nakazato [9]. We will make use of the generating function of $Q_q(\infty)$ later in our analysis.

We close this section by noting the following well-known fact, which will be used in many places throughout our study. Letting $\boldsymbol{\pi} := [\pi_n]_{n\geq 0}$ denote the PMF of $Q(\infty)$, i.e. $\pi_n := \mathbb{P}(Q(\infty) = n)$ for each integer $n \geq 0$, it follows that

$$\mathbb{P}(Q_q(\infty) = n) = \pi_0 \mathbf{1}_{\{n=0\}} + \pi_{n+1}$$

for each integer $n \geq 0$. Moreover, we can further deduce from (5) that for each integer $n \geq 0$,

$$\mathbb{P}(Q_q(\infty) = n) = \frac{\phi_{Q_q}^{(n)}(0)}{n!} = \mathbb{E}\left[\frac{(\lambda W(\infty))^n}{n!} e^{-\lambda W(\infty)}\right]$$

meaning that for each integer $n \geq 0$,

$$\mathbb{E}\left[\frac{(\lambda W(\infty))^n}{n!} e^{-\lambda W(\infty)}\right] = \pi_0 \mathbf{1}_{\{n=0\}} + \pi_{n+1}.$$

3. Time-Dependent Results

Our first objective is to derive a double transform associated with the random variables $Q(t)$, $A_{1,q}(t)$, $A_{1,s}(t)$, and $R_{1,s}(t)$, given $Q(0) = n_0$, for each real number $t > 0$. More particularly, we will derive the function $\phi_{Q,A_{1,q},A_{1,s},R_{1,s}} : \mathbb{C}_+ \times \mathbb{D} \times \mathbb{C}_+ \times \mathbb{C}_+ \times \mathbb{C}_+ \to \mathbb{C}$ defined as

$$\phi_{Q,A_{1,q},A_{1,s},R_{1,s}}(\alpha, z, \gamma_1, \gamma_2, \gamma_3) := \int_0^\infty e^{-\alpha t}\mathbb{E}_{n_0}[z^{Q(t)}e^{-(\gamma_1 A_{1,q}(t)+\gamma_2 A_{1,s}(t)+\gamma_3 R_{1,s}(t))}]dt$$

for each $z \in \mathbb{D}$, and each $\alpha, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}_+$. Theorem 3.1 given below provides an expression for $\phi_{Q,A_{1,q},A_{1,s},R_{1,s}}(\alpha, z, \gamma_1, \gamma_2, \gamma_3)$.

We will refrain from considering the random variables $A_{k,q}(t)$ and $R_{k,q}(t)$ for integers $k \geq 2$ for the moment, as introducing these random variables will make the resulting formulas (but not necessarily the analysis itself) much more complicated, as the reader will discover in the next section when we study the system in equilibrium. These random variables will be considered (in equilibrium) in the next section.

Theorem 3.1. *For each* $(\alpha, z, \gamma_1, \gamma_2, \gamma_3) \in \mathbb{C}_+ \times \mathbb{D} \times \mathbb{C}_+ \times \mathbb{C}_+ \times \mathbb{C}_+$, *we have*

$$\begin{aligned}
&\phi_{Q,A_{1,q},A_{1,s},R_{1,s}}(\alpha, z, \gamma_1, \gamma_2, \gamma_3) \qquad (6)\\
&= \frac{\pi(\alpha)^{n_0}}{\alpha + \lambda(1-\pi(\alpha))} + z\left[\frac{\beta(\gamma_3) - \beta(\alpha+\gamma_2+\lambda(1-z))}{\alpha+\gamma_2+\lambda(1-z)-\gamma_3}\right]\left[\frac{\beta(\alpha+\gamma_1+\lambda(1-z))^{n_0} - z^{n_0}}{\beta(\alpha+\gamma_1+\lambda(1-z)) - z}\right]\\
&\quad + \frac{\lambda z\pi(\alpha)}{\alpha+\lambda(1-\pi(\alpha))}\left[\frac{\beta(\gamma_3) - \beta(\alpha+\gamma_2+\lambda(1-z))}{\alpha+\gamma_2+\lambda(1-z)-\gamma_3}\right]\left[\frac{\lambda(z-\pi(\alpha)) - \gamma_1}{\lambda(z-\beta(\alpha+\gamma_1+\lambda(1-z))) - \gamma_1}\right]\\
&\quad + \lambda z\left[\frac{\beta(\gamma_3) - \beta(\alpha+\gamma_2+\lambda(1-z))}{\alpha+\gamma_2+\lambda(1-z)-\gamma_3}\right]\left[\frac{\beta(\alpha+\gamma_1+\lambda(1-z))^{n_0} - \pi(\alpha)^{n_0}}{\lambda(z-\beta(\alpha+\gamma_1+\lambda(1-z))) - \gamma_1}\right]. \qquad (7)
\end{aligned}$$

This result can be established by using an argument very similar to that used to establish versions of the time-dependent distributional Little's law (i.e. the transient distributional Little's law), which was first introduced in Bertsimas and Mourtzinou [8], although here we will modify the statement of the law slightly in order to 'zero in' on precisely when the customer currently in service at time t arrived to the system. We recommend readers also consult [16] where the time-dependent distributional Little's law is stated within the context of Palm distributions, in order to avoid having to assume the existence of certain limits. Readers should also consult Bertsimas and Gamarnik [7], a recently-published textbook on queueing theory that also discusses the time-dependent Little's law.

Proof. For each integer $k \in \{0, 1, 2, \ldots, n_0\}$, define

$$B_{1:k}^{(0)} := \sum_{\ell=1}^{k} B_\ell^{(0)}$$

where $B_{1:0}^{(0)} := 0$ (following the usual convention where empty sums are treated at being equal to zero). Our first objective is to verify that for each real $t \geq 0$,

$$z^{Q(t)}e^{-(\gamma_1 A_{1,q}(t)+\gamma_2 A_{1,s}(t)+\gamma_3 R_{1,s}(t))}$$

$$
\begin{aligned}
&= \mathbf{1}_{\{Q(t)=0\}} \\
&+ \sum_{k=1}^{n_0} \mathbf{1}_{\{B^{(0)}_{1:(k-1)} \le t, B^{(0)}_{1:(k-1)} + B^{(0)}_k > t\}} z^{n_0-k+1} z^{N(t)} e^{-(\gamma_1 B^{(0)}_{1:(k-1)} + \gamma_2 (t - B^{(0)}_{1:(k-1)}) + \gamma_3 (B^{(0)}_{1:(k-1)} + B^{(0)}_k - t))} \\
&+ \int_{(0,t]} \mathbf{1}_{\{W(s-) \le t-s, W(s-) + B_{N(s)} > t-s\}} z^{1+N(s,t]} e^{-(\gamma_1 W(s-) + \gamma_2 (t-s-W(s-)) + \gamma_3 (W(s-) + B_{N(s)} - (t-s)))} N(ds)
\end{aligned} \tag{8}
$$

where for each real number $s > 0$, $W(s-)$ is the left-hand-limit of W at time s, i.e.

$$W(s-) := \lim_{u \uparrow s} W(u).$$

Equality (8) follows as a consequence of the *overtake-free* property of this queueing system, in that people depart from the system in the same order at which they arrive. Hence, at time t, if there is a customer currently receiving service, all of the customers waiting in the queue at time t arrived after that customer.

Having this observation in mind, we can prove Equation (8) by showing that it holds when (i) the system is empty at time t; (ii) when the customer in service at time t was initially present in the system at time 0, and (iii) when the customer in service at time t arrived at some time $s \in (0, t]$.

First (i) assume the system is empty at time t. Then

$$Q(t) = A_{1,q}(t) = A_{1,s}(t) = R_{1,s}(t) = 0$$

which in turn means

$$z^{Q(t)} e^{-(\gamma_1 A_{1,q}(t) + \gamma_2 A_{1,s}(t) + \gamma_3 R_{1,s}(t))} = 1 = \mathbf{1}_{\{Q(t)=0\}}.$$

Second (ii) suppose there is a customer receiving service at time t, and that customer was present in the system at time zero. Assuming that customer was originally in slot k at time 0, it must be true that $B^{(0)}_{1:(k-1)} \le t$, $B^{(0)}_{1:(k-1)} + B^{(0)}_k > t$, and

$$A_{1,q}(t) = B^{(0)}_{1:(k-1)}, \quad A_{1,s}(t) = t - B^{(0)}_{1:(k-1)}, \quad R_{1,s}(t) = B^{(0)}_{1:(k-1)} + B^{(0)}_k - t$$

with $Q(t) = n_0 - k + 1 + N(0, t]$, because if the customer originally present in slot k at time 0 is receiving service at time t, the $n - k$ customers originally in slots $k + 1, \ldots, n$ are still present, as well as anyone else arriving in $(0, t]$. In other words,

$$
\begin{aligned}
&e^{-(\gamma_1 A_{1,q}(t) + \gamma_2 A_{1,s}(t) + \gamma_3 R_{1,s}(t))} \\
&= \mathbf{1}_{\{B^{(0)}_{1:(k-1)} \le t, B^{(0)}_{1:(k-1)} + B^{(0)}_k > t\}} e^{-(\gamma_1 B^{(0)}_{1:(k-1)} + \gamma_2 (t - B^{(0)}_{1:(k-1)}) + \gamma_3 (B^{(0)}_{1:(k-1)} + B^{(0)}_k - t))} z^{n_0-k+1} z^{N(0,t]}.
\end{aligned}
$$

Finally (iii) suppose there is a customer receiving service at time t, and that customer arrived to the system at some point $s \in (0, t]$. The amount of time this customer waits in the queue before ever receiving any attention from the server is $A_{1,q}(t) = W(s-)$, so in order for this customer to be receiving attention from the server at time t, it must be the case that $W(s-) \leq t - s$ and $W(s-) + B_{N(s)} > t - s$. Furthermore, the amount of time it has spent receiving service at time t is $A_{1,s}(t) = (t-s) - W(s-)$ (the time it has spent in the system, minus the time it spent waiting) and the remaining amount of work it possesses at time t is $R_{1,s}(t) = W(s-) + B_{N(s)} - (t-s)$ (the waiting time plus the amount of work it brought to the system, minus how long it has already spent in the system). Not only that, $Q(t) = 1 + N(s,t]$, which represents the customer who arrived at time s plus all of the arrivals that occur in $(s,t]$, and so

$$
\begin{aligned}
&z^{Q(t)} e^{-(\gamma_1 A_{1,q}(t) + \gamma_2 A_{1,s}(t) + \gamma_3 R_1(t))} \\
&= \mathbf{1}_{\{W(s-)\leq t-s, W(s-)+B_{N(s)}>t-s\}} e^{-(\gamma_1 W(s-) + \gamma_2(t-s-W(s-)) + \gamma_3(W(s-)+B_{N(s)}-(t-s)))} z^{1+N(s,t]}.
\end{aligned}
$$

Proceeding with (8), after taking the expected value of both sides, while further applying both the Campbell-Mecke formula and the Slivnyak-Mecke formula reveals that

$$
\begin{aligned}
&\mathbb{E}_{n_0}[z^{Q(t)} e^{-(\gamma_1 A_{1,q}(t) + \gamma_2 A_{1,s}(t) + \gamma_3 R_{1,s}(t))}] \\
&= \mathbb{P}_{n_0}(Q(t) = 0) \\
&\quad + \sum_{k=1}^{n_0} \mathbb{E}_{n_0}\left[\mathbf{1}_{\{B^{(0)}_{1:(k-1)} \leq t, B^{(0)}_{1:(k-1)} + B^{(0)}_k > t\}} e^{-(\gamma_1 B^{(0)}_{1:(k-1)} + \gamma_2(t - B^{(0)}_{1:(k-1)}) + \gamma_3(B^{(0)}_{1:(k-1)} + B^{(0)}_k - t))}\right] \\
&\qquad z^{n_0-k+1} e^{-\lambda(1-z)t} \\
&\quad + \lambda \int_0^t \mathbb{E}_{n_0}\left[\mathbf{1}_{\{W(s-)\leq t-s, W(s-)+B_{N(s)}>t-s\}} e^{-(\gamma_1 W(s-) + \gamma_2(t-s-W(s-)) + \gamma_3(W(s-)+B_{N(s)}-(t-s)))}\right] \\
&\qquad z e^{-\lambda(1-z)(t-s)} ds
\end{aligned}
\tag{9}
$$

where B denotes a generic random variable having CDF F and independent of $W(s)$. After multiplying both sides of (9) by $e^{-\alpha t}$, then integrating both sides over $[0, \infty)$ with respect to t, we get

$$
\begin{aligned}
&\phi_{Q,A_{1,q},A_{1,s},R_{1,s}}(\alpha, z, \gamma_1, \gamma_2, \gamma_3) \\
&= \phi_{n_0;0}(\alpha) \\
&\quad + \sum_{k=1}^{n_0} \mathbb{E}\left[e^{-\gamma_1 B^{(0)}_{1:(k-1)}} \int_{B^{(0)}_{1:(k-1)}}^{B^{(0)}_{1:(k-1)} + B^{(0)}_k} e^{-\gamma_2(t - B^{(0)}_{1:(k-1)})} e^{-\gamma_3(B^{(0)}_{1:(k-1)} + B^{(0)}_k - t)} e^{-(\alpha + \lambda(1-z))t} dt\right] z^{n_0-k+1} \\
&\quad + \lambda z \int_0^\infty \mathbb{E}\left[e^{-\gamma_1 W(s)} \int_{W(s)}^{W(s)+B} e^{-\gamma_2(t-W(s))} e^{-\gamma_3(B-(t-W(s)))} e^{-(\alpha+\lambda(1-z))t} dt\right] e^{-\alpha s} ds \\
&= \phi_{n_0;0}(\alpha) \\
&\quad + \sum_{k=1}^{n_0} \mathbb{E}\left[e^{-(\alpha+\lambda(1-z)+\gamma_1) B^{(0)}_{1:(k-1)}} \int_0^{B^{(0)}_k} e^{-(\gamma_2+\alpha+\lambda(1-z))y} e^{-\gamma_3(B^{(0)}_k - y)} dy\right] z^{n_0-(k-1)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda z \int_0^\infty \mathbb{E}_{n_0}\left[e^{-(\alpha+\lambda(1-z)+\gamma_1)W(s)} \int_0^B e^{-(\gamma_2+\alpha+\lambda(1-z))y} e^{-\gamma_3(B-y)} dy\right] e^{-\alpha s} ds \\
&= \phi_{n_0;0}(\alpha) + \left[\frac{\beta(\gamma_3) - \beta(\alpha+\gamma_2+\lambda(1-z))}{\alpha+\gamma_2+\lambda(1-z)-\gamma_3}\right] \sum_{k=1}^{n_0} \beta(\alpha+\gamma_1+\lambda(1-z))^{k-1} z^{n_0-(k-1)} \\
&\quad + \lambda z \left[\frac{\beta(\gamma_3) - \beta(\alpha+\gamma_2+\lambda(1-z))}{\alpha+\gamma_2+\lambda(1-z)-\gamma_3}\right] \varphi_{n_0;W}(\alpha, \alpha+\gamma_1+\lambda(1-z)) \\
&= \frac{\pi(\alpha)^{n_0}}{\alpha+\lambda(1-\pi(\alpha))} + z \left[\frac{\beta(\gamma_3) - \beta(\alpha+\gamma_2+\lambda(1-z))}{\alpha+\gamma_2+\lambda(1-z)-\gamma_3}\right] \left[\frac{\beta(\alpha+\gamma_1+\lambda(1-z))^{n_0} - z^{n_0}}{\beta(\alpha+\gamma_1+\lambda(1-z)) - z}\right] \\
&\quad + \frac{\lambda z \pi(\alpha)}{\alpha+\lambda(1-\pi(\alpha))} \left[\frac{\beta(\gamma_3) - \beta(\alpha+\gamma_2+\lambda(1-z))}{\alpha+\gamma_2+\lambda(1-z)-\gamma_3}\right] \left[\frac{\lambda(z-\pi(\alpha)) - \gamma_1}{\lambda(z-\beta(\alpha+\gamma_1+\lambda(1-z))) - \gamma_1}\right] \\
&\quad + \lambda z \left[\frac{\beta(\gamma_3) - \beta(\alpha+\gamma_2+\lambda(1-z))}{\alpha+\gamma_2+\lambda(1-z)-\gamma_3}\right] \left[\frac{\beta(\alpha+\gamma_1+\lambda(1-z))^{n_0} - \pi(\alpha)^{n_0}}{\lambda(z-\beta(\alpha+\gamma_1+\lambda(1-z))) - \gamma_1}\right]. \qquad (10)
\end{aligned}
$$

This completes the proof of the claim.

Before we proceed further, it is worth looking at various special cases of this result. For instance, if we further assume $\rho < 1$ so that the system approaches equilibrium as $t \to \infty$, we can use Theorem 3.1 to study the joint distribution of the random vector

$$
(Q(\infty), A_{1,q}(\infty), A_{1,s}(\infty), R_{1,s}(\infty))
$$

where $Q(\infty)$ represents the number of customers in the system in equilibrium, $A_{1,q}(\infty)$ represents the amount of time the customer currently in service spent waiting in the queue (if there is such a customer) in equilibrium, $A_{1,s}(\infty)$ represents the amount of time the customer currently in service has been in service in equilibrium, and $R_{1,s}(\infty)$ represents the remaining amount of time the customer in service will spend in service, again in equilibrium. Standard arguments from regenerative process theory can be used to show that this joint distribution exists and can be interpreted as a limiting distribution, and moreover, for each $z \in \mathbb{D}$ and each $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}_+$,

$$
\mathbb{E}[z^{Q(\infty)} e^{-(\gamma_1 A_{1,q}(\infty)+\gamma_2 A_{1,s}(\infty)+\gamma_3 R_{1,s}(\infty))}] = \lim_{\alpha \downarrow 0} \alpha \phi_{Q,A_{1,q},A_{1,s},R_{1,s}}(\alpha, z, \gamma_1, \gamma_2, \gamma_3).
$$

Corollary 3.1. *The Laplace-Stieltjes transform of* $Q(\infty), A_{1,q}(\infty), A_{1,s}(\infty), R_{1,s}(\infty)$ *is as follows:*

$$
\begin{aligned}
&\mathbb{E}[z^{Q(\infty)} e^{-(\gamma_1 A_{1,q}(\infty)+\gamma_2 A_{1,s}(\infty)+\gamma_3 R_1(\infty))}] \\
&= (1-\rho) + \lambda(1-\rho) z \left[\frac{\beta(\gamma_3) - \beta(\gamma_2+\lambda(1-z))}{\gamma_2+\lambda(1-z)-\gamma_3}\right] \left[\frac{\lambda(z-1) - \gamma_1}{\lambda(z-\beta(\gamma_1+\lambda(1-z))) - \gamma_1}\right].
\end{aligned}
$$

4. More Refined Results in Equilibrium

Our next objective is to study the conditional joint distribution, given the steady-state queue-length is equal to n, of how long the customer currently in service waited in line before receiving service, how long that customer has spent in service, and the remaining amount of time that customer will spend in service. The approach we will use to study this system—once it has reached equilibrium—requires us to construct the M/G/1 queue on all of $\mathbb{R}$ instead of only on $[0,\infty)$, but fortunately this is possible, and doing so results in a system that is in equilibrium (readers interested in further details should consult Chapters 1 and 2 of Baccelli and Brémaud [6], we refrain from providing a description of shift operators and shift-invariance here because the details are not necessary to understand what follows). We can model the system in equilibrium by constructing a homogeneous Poisson process N defined on the entire real line $\mathbb{R}$, having points $\{T_n\}_{n\in\mathbb{Z}}$, where these points satisfy, with probability one,

$$\cdots < T_{-2} < T_{-1} < T_0 \leq 0 < T_1 < T_2 < T_3 < \cdots .$$

Assume further that the customer arriving at time T_n brings an amount of work $\{B_n\}_{n\in\mathbb{Z}}$ for processing, where the sequence $\{B_n\}_{n\in\mathbb{Z}}$ is assumed to be i.i.d. with CDF F, and independent of everything else. Assuming $\rho := \lambda\mathbb{E}[B_1] < \infty$, it is well known (see e.g. Baccelli and Brémaud [6] for details) that the queue-length process $\{Q(t); t \in \mathbb{R}\}$ of the M/G/1 queue operating under the FCFS service discipline will experience an empty state infinitely often over $(-\infty, 0]$ (as well as over $[0,\infty)$ for that matter), and is strictly stationary, where the law of $Q(0)$ coincides with the limiting distribution of the number of customers in the system. The same can be said for the workload process $\{W(t); t \in \mathbb{R}\}$ of the M/G/1 queue, as well as the processes $\{A_{1,q}(t); t \in \mathbb{R}\}$, $\{R_{1,s}(t); t \in \mathbb{R}\}$, and so on.

4.1. Conditional distibution of age and residual times of the customer in service

Again, our first objective in this section is to derive an explicit formula for the conditional joint Laplace-Stieltjes transform of $A_{1,q}(0)$, $A_{1,s}(0)$, and $R_{1,s}(0)$, given $Q(0) = n$ for some fixed integer $n \geq 1$. The approach we will use to calculate this conditional joint transform will require us to make use of the following elementary lemmas.

The first lemma addresses the collection of functions $\{a_n\}_{n\geq 1}$, where for each integer $n \geq 1$, we define $a_n : [0,\infty) \times \mathbb{C}_+ \times \mathbb{C}_+ \to \mathbb{C}$ as

$$a_n(x,\gamma_1,\gamma_2) = \int_0^x e^{-\gamma_1 s} e^{-\gamma_2 (x-s)} \frac{\lambda(\lambda s)^{n-1} e^{-\lambda s}}{(n-1)!} ds.$$

Lemma 4.1. *For each $x \geq 0$, and each $\gamma_1, \gamma_2 \in \mathbb{C}_+$, we have*

$$a_n(x,\gamma_1,\gamma_2) = \left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right]^n e^{-\gamma_2 x} - \sum_{\ell=0}^{n-1} \left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right]^{n-\ell} \frac{(\lambda x)^\ell e^{-(\lambda+\gamma_1)x}}{\ell!}.$$

Proof. This result can be established using induction. First,

Next, observe that for each integer $n \geq 1$,

$$a_{n+1}(x,\gamma_1,\gamma_2) = \int_0^x e^{-\gamma_1 s} e^{-\gamma_2 (x-s)} \frac{\lambda(\lambda s)^n}{n!} e^{-\lambda s} ds$$

$$= \lambda^{n+1} e^{-\gamma_2 x} \int_0^x \frac{s^n}{n!} e^{-(\lambda+\gamma_1-\gamma_2)s} ds. \tag{11}$$

Applying integration by parts to the integral found in (11) reveals that

$$a_{n+1}(x,\gamma_1,\gamma_2) = -\lambda^{n+1} e^{-\gamma_2 x} \frac{x^n}{n!} \frac{e^{-(\lambda+\gamma_1-\gamma_2)x}}{\lambda+\gamma_1-\gamma_2} + \frac{\lambda^{n+1} e^{-\gamma_2 x}}{\lambda+\gamma_1-\gamma_2} \int_0^x \frac{s^{n-1}}{(n-1)!} e^{-(\lambda+\gamma_1-\gamma_2)s} ds$$

and this equality can alteratively be expressed as

$$a_{n+1}(x,\gamma_1,\gamma_2) = (-1)\left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right] \frac{(\lambda x)^n}{n!} e^{-(\lambda+\gamma_1)x} + \left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right] a_n(x,\gamma_1,\gamma_2).$$

From this recursion, combined with (**??**), a quick induction argument can be used to establish the claim.

The next lemma makes use of Lemma (4.1) in order to derive an expected value that will appear in our next main result.

Lemma 4.2. *For each integer $n \geq 1$, each $\gamma_1, \gamma_2 \in \mathbb{C}_+$,*

$$\mathbb{E}\left[\int_0^B e^{-\gamma_1 s} e^{-\gamma_2 (B-s)} \frac{\lambda(\lambda s)^{n-1} e^{-\lambda s}}{(n-1)!} ds\right] = \left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right]^n \beta(\gamma_2) - \sum_{\ell=0}^{n-1} \left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right]^{n-\ell} \frac{(-\lambda)^\ell}{\ell!} \beta^{(\ell)}(\lambda+\gamma_1). \tag{12}$$

If we further assume that F has a PDF f, then

$$\mathbb{E}\left[\int_0^B e^{-\gamma_1 s} e^{-\gamma_2 (B-s)} \frac{\lambda(\lambda s)^{n-1} e^{-\lambda s}}{(n-1)!} ds\right] = \int_0^\infty \int_0^\infty e^{-\gamma_1 s} e^{-\gamma_2 y} \frac{\lambda(\lambda s)^{n-1} e^{-\lambda s}}{(n-1)!} f(y+s) dy ds. \tag{13}$$

Proof. First, observe that for each integer $n \geq 1$,

$$\mathbb{E}\left[\int_0^B e^{-\gamma_1 s} e^{-\gamma_2 (B-s)} \frac{\lambda(\lambda s)^{n-1} e^{-\lambda s}}{(n-1)!} ds\right] = \mathbb{E}\left[a_n(B,\gamma_1,\gamma_2)\right]$$

and clearly

$$\begin{aligned} \mathbb{E}\left[a_n(B,\gamma_1,\gamma_2)\right] &= \mathbb{E}\left[\left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right]^n e^{-\gamma_2 B}\right] - \mathbb{E}\left[\sum_{\ell=0}^{n-1} \left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right]^{n-\ell} \frac{(\lambda x)^\ell e^{-(\lambda+\gamma_1)B}}{\ell!}\right] \\ &= \left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right]^n \beta(\gamma_2) - \sum_{\ell=0}^{n-1} \left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right]^{n-\ell} \frac{(-\lambda)^\ell}{\ell!} \beta^{(\ell)}(\lambda+\gamma_1) \end{aligned}$$

which yields (12). Establishing (13) requires conditioning: here

$$\begin{aligned}\mathbb{E}\left[\int_0^B e^{-\gamma_1 s}e^{-\gamma_2(B-s)}\frac{\lambda(\lambda s)^{n-1}e^{-\lambda s}}{(n-1)!}ds\right] &= \int_0^\infty\int_0^x e^{-\gamma_1 s}e^{\gamma_2(x-s)}\frac{\lambda(\lambda s)^{n-1}e^{-\lambda s}}{(n-1)!}f(x)dsdx\\ &= \int_0^\infty\int_s^\infty e^{-\gamma_1 s}e^{-\gamma_2(x-s)}\frac{\lambda(\lambda s)^{n-1}e^{-\lambda s}}{(n-1)!}f(x)dxds\\ &= \int_0^\infty\int_0^\infty e^{-\gamma_1 s}e^{-\gamma_2 y}\frac{\lambda(\lambda s)^{n-1}e^{-\lambda s}}{(n-1)!}f(y+s)dyds\end{aligned}$$

where the last equality follows from applying a simple change of variable.

We are now ready to state and prove our first main result. Note that for each $t\in\mathbb{R}$, $Q_q(t)$ denotes the number of customers waiting in the queue at time t, and ϕ_{Q_q} denotes its generating function, i.e.

$$\phi_{Q_q}(z) := \mathbb{E}[z^{Q_q(0)}].$$

Theorem 4.1. *For each integer $n\geq 1$, the conditional joint Laplace-Stieltjes transform of $A_{1,q}(0)$, $A_{1,s}(0)$, and $R_{1,s}(0)$, given $Q(0)=n$ is as follows:*

$$\begin{aligned}&\mathbb{E}[e^{-(\gamma_1 A_{1,q}(0)+\gamma_2 A_{1,s}(0)+\gamma_3 R_{1,s}(0))}\mid Q(0)=n] \qquad (14)\\ &= \frac{1}{\pi_n}\sum_{k=0}^{n-1}\frac{\phi^{(k)}_{Q_q(0)}(-\gamma_1/\lambda)}{k!}\left[\left[\frac{\lambda}{\lambda+\gamma_2-\gamma_3}\right]^{n-k}\beta(\gamma_3)\right.\\ &\left.\quad-\sum_{\ell=0}^{n-k-1}\left[\frac{\lambda}{\lambda+\gamma_2-\gamma_3}\right]^{n-k-\ell}\frac{(-\lambda)^\ell}{\ell!}\beta^{(\ell)}(\lambda+\gamma_2)\right].\end{aligned}$$

Proof. The proof of this result is similar to the proof technique used to establish Theorem 3.1. For each integer $n\geq 1$,

$$\begin{aligned}&\mathbf{1}_{\{Q(0)=n\}}e^{-(\gamma_1 A_{1,q}(0)+\gamma_2 A_{1,s}(0)+\gamma_3 R_{1,s}(0))}\\ &=\int_{(-\infty,0]}\mathbf{1}_{\{W(s-)\leq -s, W(s-)+B_{N[-s,0]+1}>-s\}}e^{-\gamma_1 W(s-)}e^{-\gamma_2((-s)-W(s-))}e^{-\gamma_3(W(s-)+B_{N[-s,0]+1}-(-s))}\\ &\qquad\mathbf{1}_{\{N(-s,0]=n-1\}}N(ds).\end{aligned}$$

After taking the expected value of both sides, then applying both the Campbell-Mecke formula and the Slivnyak-Mecke formula to the right-hand-side, as well as the fact that the workload process is strictly stationary, we get

$$\begin{aligned}&\mathbb{E}[\mathbf{1}_{\{Q(0)=n\}}e^{-(\gamma_1 A_{1,q}(0)+\gamma_2 A_{1,s}(0)+\gamma_3 R_{1,s}(0))}]\\ &=\lambda\int_{-\infty}^0\mathbb{E}\left[\mathbf{1}_{\{W(0)\leq -s, W(0)+B>-s\}}e^{-\gamma_1 W(0)}e^{-\gamma_2((-s)-W(0))}e^{-\gamma_3(W(0)+B-(-s))}\right]\frac{(-\lambda s)^{n-1}e^{-\lambda(-s)}}{(n-1)!}ds\end{aligned}$$

$$
\begin{aligned}
&= \lambda \int_0^\infty \mathbb{E}\left[\mathbf{1}_{\{W(0)\le s, W(0)+B>s\}} e^{-\gamma_1 W(0)} e^{-\gamma_2(s-W(0))} e^{-\gamma_3(W(0)+B-s)}\right] \frac{(\lambda s)^{n-1}e^{-\lambda s}}{(n-1)!} ds \\
&= \lambda \mathbb{E}\left[e^{-\gamma_1 W(0)} \int_{W(0)}^{W(0)+B} e^{-\gamma_2(s-W(0))} e^{-\gamma_3(B-(s-W(0)))} \frac{(\lambda s)^{n-1}e^{-\lambda s}}{(n-1)!} ds\right] \\
&= \sum_{k=0}^{n-1} \mathbb{E}\left[e^{-(\lambda+\gamma_1)W(0)} \frac{(\lambda W(0))^k}{k!}\right] \mathbb{E}\left[\int_0^B e^{-\gamma_2 s} e^{-\gamma_3(B-s)} \frac{\lambda(\lambda s)^{n-1-k}e^{-\lambda s}}{(n-1-k)!} ds\right]. \qquad (15)
\end{aligned}
$$

Furthermore, we know from the distributional Little's law that

$$
\mathbb{E}[z^{Q_q(0)}] = \mathbb{E}[e^{-\lambda(1-z)W(0)}]
$$

which further implies that for each integer $\ell \ge 1$,

$$
\phi_{Q_q}^{(\ell)}(z) = \mathbb{E}[(\lambda W(0))^\ell e^{-\lambda(1-z)W(0)}]
$$

which clearly exists for all $z \in \mathbb{C}$ satisfying $Re(z) < 1$, and moreover,

$$
\phi_{Q_q}^{(\ell)}(-\gamma_1/\lambda) = \mathbb{E}\left[(\lambda W(0))^\ell e^{-(\lambda+\gamma_1)W(0)}\right].
$$

After applying both this observation and (12) to (15), then simplifying, we arrive at the claim.

Many of the results from [4] follow as a consequence of Theorem 4.1, as the following corollary illustrates.

Corollary 4.1. *For each integer $n \ge 1$, and each $\gamma_1, \gamma_2 \in \mathbb{C}_+$,*

$$
\begin{aligned}
&\mathbb{E}[e^{-(\gamma_1 A_{1,s}(0)+\gamma_2 R_{1,s}(0))} \mid Q(0) = n] \qquad (16) \\
&= \frac{\pi_0}{\pi_n}\left[\left(\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right)^n \beta(\gamma_2) - \sum_{\ell=0}^{n-1}\left(\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right)^{n-\ell} \frac{(-\lambda)^\ell}{\ell!}\beta^{(\ell)}(\lambda+\gamma_1)\right] \\
&\quad + \sum_{k=0}^{n-1}\frac{\pi_{k+1}}{\pi_n}\left[\left(\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right)^{n-k}\beta(\gamma_2) - \sum_{\ell=0}^{n-(k+1)}\left(\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right)^{n-k-\ell}\frac{(-\lambda)^\ell}{\ell!}\beta^{(\ell)}(\lambda+\gamma_1)\right].
\end{aligned}
$$

Moreover, when the service time distribution F is absolutely continuous with PDF f, the conditional distribution of $A_{1,s}(0)$ and $R_{1,s}(0)$ given $Q(0) = n$ has a joint PDF of the form

$$
\begin{aligned}
f_{A_{1,s}(0),R_{1,s}(0)|Q(0)=n}(s,y) =& \frac{\pi_0}{\pi_n}\frac{\lambda(\lambda s)^{n-1}e^{-\lambda s}}{(n-1)!} f(s+y) \\
&+ \sum_{k=1}^{n}\frac{\pi_k}{\pi_n}\frac{\lambda(\lambda s)^{n-k}e^{-\lambda s}}{(n-k)!} f(s+y), \ s, y > 0. \qquad (17)
\end{aligned}
$$

Proof. Formula (16) is a special case of (14) in Theorem 4.1. In order to establish (17), observe first from our proof of Theorem 4.1 that for each $\gamma_1, \gamma_2 \in \mathbb{C}_+$,

$$\begin{aligned}&\mathbb{E}[\mathbf{1}_{\{Q(0)=n\}} e^{-(\gamma_1 A_{1,s}(0) + \gamma_2 R_{1,s}(0))}] \\ &= \sum_{k=0}^{n-1} (\pi_0 \mathbf{1}_{\{k=0\}} + \pi_{k+1}) \mathbb{E}\left[\int_0^B e^{-\gamma_1 s} e^{-\gamma_2 (B-s)} \frac{\lambda(\lambda s)^{n-k-1} e^{-\lambda s}}{(n-k-1)!} ds\right]\end{aligned}$$

and applying (13) to this expression yields

$$\begin{aligned}&\mathbb{E}[\mathbf{1}_{\{Q(0)=n\}} e^{-(\gamma_1 A_{1,s}(0) + \gamma_2 R_{1,s}(0))}] \\ &= \sum_{k=0}^{n-1} (\pi_0 \mathbf{1}_{\{k=0\}} + \pi_{k+1}) \int_0^\infty \int_0^\infty e^{-\gamma_1 s} e^{-\gamma_2 y} \frac{\lambda(\lambda s)^{n-k-1} e^{-\lambda s}}{(n-k-1)!} f(y+s) dy ds.\end{aligned}$$

From this formula, we quickly observe that the conditional joint density of $A_{1,s}(0)$ and $R_{1,s}(0)$, given $Q(0) = n$ indeed exists.

It should be noted that this conditional joint density appears (somewhat implicitly) in Equation (24) on page 121 of [4], as they derive it from the conditional PDF of $A_{1,s}(0)$ given $Q(0) = n$, while further noting that given $A_{1,s}(0)$ and $Q(0) \geq 1$, $R_{1,s}(0)$ and $Q(0)$ are independent.

4.2. Conditional age and residual moments of the customer in service

We next turn our attention to deriving various conditional moments of $A_{1,q}(0)$, $A_{1,s}(0)$, and $R_{1,s}(0)$, given $Q(0) = n$ where n is a fixed positive integer. In Sigman and Yechiali [24], the authors used a rate conservation law to derive a recursion satisfied by the conditional kth moment of the residual service time, given $Q(0) = n$ for each integer $n \geq 1$. We use an alternative approach, and in the process of doing so we show how these conditional moments, as well as other moments that involve both $A_{1,q}(0)$ and $A_{1,s}(0)$ can be stated more explicitly using residual distributions.

Recall that given a nonnegative random variable B having CDF F, the residual of B is a nonnegative random variable $R_{1,B}$ whose CDF is given by $F_{(1,e)}$, where for each $t \geq 0$,

$$F_{(1,e)}(t) := \frac{1}{\mathbb{E}[B]} \int_0^t \mathbb{P}(B > s) ds.$$

We use the notation $(1, e)$ in $F_{(1,e)}$ as sometimes this CDF is referred to as the equilibrium CDF associated with F. The LST of $R_{1,B}$ is $\beta_{(1,e)}$, which satisfies

$$\beta_{(1,e)}(\alpha) = \frac{1 - \beta(\alpha)}{\alpha \mathbb{E}[B]}$$

for each $\alpha \in \mathbb{C}_+$. We can also speak of 'residuals of residuals' in that for each integer $m \geq 1$, $R_{m+1,B}$ is simply the residual of $R_{m,B}$, where $R_{m,B}$ has CDF $F_{(m,e)}$ and LST $\beta_{(m,e)}$. Properties of these residual distributions are covered in the Appendix.

The following lemma will be used in the derivation of our next main result.

Lemma 4.3. *Let m_1, m_2 be nonnegative integers. Then for each integer $n \geq 1$,*

$$\mathbb{E}\left[\int_0^B s^{m_1}(B-s)^{m_2}\frac{\lambda(\lambda s)^{n-1}e^{-\lambda s}}{(n-1)!}ds\right] = \frac{\lambda\mathbb{E}[B^{m_2+1}]}{(m_2+1)}\frac{(-1)^{m_1}(-\lambda)^{n-1}}{(n-1)!}\beta^{(m_1+n-1)}_{(m_2+1,e)}(\lambda). \tag{18}$$

Proof. First, observe that from Proposition A.2,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^B s^{m_1}(B-s)^{m_2}\frac{\lambda(\lambda s)^{n-1}e^{-\lambda s}}{(n-1)!}ds\right] \\
&= \int_0^\infty s^{m_1}\mathbb{E}[(B-s)^{m_2}\mathbf{1}_{\{B>s\}}]\frac{\lambda(\lambda s)^{n-1}e^{-\lambda s}}{(n-1)!}ds \\
&= \lambda\mathbb{E}[B^{m_2}]\frac{(-1)^{m_1}(-\lambda)^{n-1}}{(n-1)!}\int_0^\infty(-s)^{m_1+n-1}e^{-\lambda s}\mathbb{P}(R_{m_2,B}>s)ds \\
&= \lambda\mathbb{E}[B^{m_2}]\mathbb{E}[R_{m_2,B}]\frac{(-1)^{m_1}(-\lambda)^{n-1}}{(n-1)!}\beta^{(m_1+n-1)}_{(m_2+1,e)}(\lambda).
\end{aligned}$$

Using Proposition A.1, we get

$$\mathbb{E}[B^{m_2}]\mathbb{E}[R_{m_2,B}] = \mathbb{E}[B^{m_2}]\frac{\mathbb{E}[B^{1+m_2}]}{\binom{1+m_2}{m_2}\mathbb{E}[B^{m_2}]} = \frac{\mathbb{E}[B^{m_2+1}]}{(m_2+1)}$$

which means

$$\mathbb{E}\left[\int_0^B s^{m_1}(B-s)^{m_2}\frac{\lambda(\lambda s)^{n-1}e^{-\lambda s}}{(n-1)!}ds\right] = \frac{\lambda\mathbb{E}[B^{m_2+1}]}{(m_2+1)}\frac{(-1)^{m_1}(-\lambda)^{n-1}}{(n-1)!}\beta^{(m_1+n-1)}_{(m_2+1,e)}(\lambda)$$

which proves the claim.

Theorem 4.2. *For each integer $n \geq 1$, the following statements are true: (a) for each pair of positive integers m_2 and m_3,*

$$\begin{aligned}
\mathbb{E}[A_{1,s}(0)^{m_2}R_{1,s}(0)^{m_3} \mid Q(0)=n] &= \frac{\pi_0}{\pi_n}\frac{\lambda\mathbb{E}[B^{m_3+1}]}{(m_3+1)}\frac{(-1)^{m_2}(-\lambda)^{n-1}}{(n-1)!}\beta^{(m_2+n-1)}_{(m_3+1,e)}(\lambda) \quad (19)\\
&+\sum_{k=0}^{n-1}\frac{\pi_{k+1}}{\pi_n}\frac{\lambda\mathbb{E}[B^{m_3+1}]}{(m_3+1)}\frac{(-1)^{m_2}(-\lambda)^{n-k-1}}{(n-k-1)!}\beta^{(m_2+n-k-1)}_{(m_3+1,e)}(\lambda).
\end{aligned}$$

Moreover (b) for each triplet of positive integers m_1, m_2, and m_3,

$$\begin{aligned}
&\mathbb{E}[A_{1,q}(0)^{m_1}A_{1,s}(0)^{m_2}R_{1,s}(0)^{m_3} \mid Q(0)=n] \quad (20)\\
&= \sum_{k=0}^{n-1}\frac{(m_1+k)!}{k!}\frac{\pi_{m_1+k+1}}{\pi_n}\frac{\lambda\mathbb{E}[B^{m_3+1}]}{(m_3+1)}\frac{(-1)^{m_2}(-\lambda)^{n-k-1}}{(n-k-1)!}\beta^{(m_2+n-k-1)}_{(m_3+1,e)}(\lambda).
\end{aligned}$$

Proof. For each integer $n \geq 0$, and for each set of integers $m_1, m_2, m_3 \geq 0$,

$$
\begin{aligned}
&\mathbf{1}_{\{Q(0)=n\}} A_{1,q}(0)^{m_1} A_{1,s}(0)^{m_2} R_{1,s}(0)^{m_3} \\
&= \int_{(-\infty,0]} \mathbf{1}_{\{W(s-)\leq -s, W(s-)+B_{-N[-s,0]+1}>-s\}} W(s-)^{m_1} \\
&\times ((-s) - W(s-))^{m_2} (W(s-) + B_{-N[-s,0]+1} - (-s))^{m_3} \mathbf{1}_{\{N(-s,0]=n-1\}} N(ds).
\end{aligned}
$$

After taking the expected value of both sides, then applying both the Campbell-Mecke formula and the Slivnyak-Mecke Theorem to the right-hand-side, we get

$$
\begin{aligned}
&\mathbb{E}[\mathbf{1}_{\{Q(0)=n\}} A_{1,q}(0)^{m_1} A_{1,s}(0)^{m_2} R_{1,s}(0)^{m_3}] \\
&= \lambda \int_{-\infty}^{0} \mathbb{E}[\mathbf{1}_{\{W(0)\leq -s, W(0)+B>-s\}} W(0)^{m_1} ((-s) - W(0))^{m_2} (W(0) + B - (-s))^{m_3}] \\
&\qquad \frac{(-\lambda s)^{n-1} e^{-\lambda(-s)}}{(n-1)!} ds \\
&= \lambda \int_{0}^{\infty} \mathbb{E}[\mathbf{1}_{\{W(0)\leq s, W(0)+B>s\}} W(0)^{m_1} (s - W(0))^{m_2} (W(0) + B - s)^{m_3}] \frac{(\lambda s)^{n-1} e^{-\lambda s}}{(n-1)!} ds \\
&= \lambda \mathbb{E}\left[\int_{W(0)}^{W(0)+B} W(0)^{m_1} (s - W(0))^{m_2} (W(0) + B - s)^{m_3} \frac{(\lambda s)^{n-1} e^{-\lambda s}}{(n-1)!} ds \right] \\
&= \lambda \mathbb{E}\left[W(0)^{m_1} \int_{0}^{B} y^{m_2} (B - y)^{m_3} \frac{(\lambda(W(0)+y))^{n-1} e^{-\lambda(W(0)+y)}}{(n-1)!} dy \right] \\
&= \frac{1}{\lambda^{m_1}} \sum_{k=0}^{n-1} \mathbb{E}\left[(\lambda W(0))^{m_1+k} \frac{e^{-\lambda W(0)}}{k!} \right] \mathbb{E}\left[\int_{0}^{B} y^{m_2} (B - y)^{m_3} \frac{\lambda (\lambda y)^{n-(k+1)} e^{-\lambda y}}{(n-(k+1))!} dy \right].
\end{aligned}
$$

Next, observe that since

$$
\mathbb{E}\left[(\lambda W(0))^{m_1+k} \frac{e^{-\lambda W(0)}}{k!} \right] = \frac{(m_1+k)!}{k!} \left[\pi_0 \mathbf{1}_{\{m_1+k=0\}} + \pi_{m_1+k+1} \right]
$$

and

$$
\mathbb{E}\left[\int_{0}^{B} y^{m_2} (B - y)^{m_3} \frac{\lambda (\lambda y)^{n-(k+1)} e^{-\lambda y}}{(n-(k+1))!} dy \right] = \frac{\lambda \mathbb{E}[B^{m_3+1}]}{(m_3+1)} \frac{(-1)^{m_2} (-\lambda)^{n-k-1}}{(n-k-1)!} \beta_{(m_3+1,e)}^{(m_2+n-k-1)}(\lambda)
$$

we get

$$
\begin{aligned}
&\mathbb{E}[\mathbf{1}_{\{Q(0)=n\}} A_{1,q}(0)^{m_1} A_{1,s}(0)^{m_2} R_{1,s}(0)^{m_3}] \\
&= \frac{1}{\lambda^{m_1}} \sum_{k=0}^{n-1} \frac{(m_1+k)!}{k!} (\pi_0 \mathbf{1}_{\{m_1+k=0\}} + \pi_{m_1+k+1}) \frac{\lambda \mathbb{E}[B^{m_3+1}]}{(m_3+1)} \frac{(-1)^{m_2} (-\lambda)^{n-k-1}}{(n-k-1)!} \beta_{(m_3+1,e)}^{(m_2+n-k-1)}(\lambda).
\end{aligned}
$$

From this formula, it is clear that for each $m_2 \geq 0$ and each $m_3 \geq 0$,

$$\mathbb{E}[\mathbf{1}_{\{Q(0)=n\}}A_{1,s}(0)^{m_2}R_{1,s}(0)^{m_3}] = \pi_0 \frac{\lambda\mathbb{E}[B^{m_3+1}]}{(m_3+1)}\frac{(-1)^{m_2}(-\lambda)^{n-1}}{(n-1)!}\beta^{(m_2+n-1)}_{(m_3+1,e)}(\lambda) + \sum_{k=0}^{n-1}\pi_{k+1}\frac{\lambda\mathbb{E}[B^{m_3+1}]}{(m_3+1)}\frac{(-1)^{m_2}(-\lambda)^{n-k-1}}{(n-k-1)!}\beta^{(m_2+n-k-1)}_{(m_3+1,e)}(\lambda)$$

and for each $m_1 \geq 1$, $m_2 \geq 0$, and $m_3 \geq 0$,

$$\mathbb{E}[\mathbf{1}_{\{Q(0)=n\}}A_{1,q}(0)^{m_1}A_{1,s}(0)^{m_2}R_{1,s}(0)^{m_3}] = \sum_{k=0}^{n-1}\frac{(m_1+k)!}{k!}\pi_{m_1+k+1}\frac{\lambda\mathbb{E}[B^{m_3+1}]}{(m_3+1)}\frac{(-1)^{m_2}(-\lambda)^{n-k-1}}{(n-k-1)!}\beta^{(m_2+n-k-1)}_{(m_3+1,e)}(\lambda)$$

from which we arrive at the claim.

The following corollary shows that all of the conditional moments of $R_{1,s}(0)$, given $Q(0) = n$, can be expressed in a surprisingly explicit form. In fact, the moment expressions found in Theorem 4.2 are basically just as explicit, but we focus here on only the moments of $R_{1,s}(0)^n$ given $Q(0)$ as these have been studied by others in the past: the first conditional moment of $R_{1,s}(0)$, given $Q(0) = n$ is studied in both [21, 13], and [24] explain how higher moments can be calculated recursively (but no explicit expression for these moments is given there).

Corollary 4.2. *For each integer $n \geq 1$, and each integer $m \geq 1$,*

$$\mathbb{E}[R_{1,s}(0)^m \mid Q(0) = n] = \frac{\lambda\mathbb{E}[B^{m+1}]}{(m+1)}\left[\frac{\pi_0}{\pi_n}\frac{(-\lambda)^{n-1}}{(n-1)!}\beta^{(n-1)}_{(m+1,e)}(\lambda) + \sum_{k=1}^{n}\frac{\pi_k}{\pi_n}\frac{(-\lambda)^{n-k}}{(n-k)!}\beta^{(n-k)}_{(m+1,e)}(\lambda)\right]$$

We conclude by showing how to develop a recursion satisfied by the derivatives of each $\beta_{(m,e)}$ transform. For each integer $m \geq 0$, and each integer $k \geq 0$, we define

$$x_{m,k}(\alpha) := \frac{(-\alpha)^k\beta^{(k)}_{(m,e)}(\alpha)}{k!}$$

then it is easy to show that these terms satisfy a simple recursive scheme.

Proposition 4.1. *For each integer $m \geq 0$, and each integer $k \geq 1$, we get*

$$x_{m+1,k}(\alpha) = x_{m+1,k-1}(\alpha) - \frac{x_{m,k}(\alpha)}{\alpha\mathbb{E}[R_{m,B}]}.$$

This recursion reveals that all $x_{m,k}(\alpha)$ terms can be calculated once the $x_{0,k}(\alpha)$ terms are known, and readers should note that these are needed in order to even calculate the transition matrix associated with the embedded discrete-time Markov chain that keeps track of the number of customers present in an M/G/1 queue (including the customer in service, if there is one) right before arrival instants.

Proof. Observe first that for each $\alpha \in \mathbb{C}_+$, and each integer $m \geq 0$,

$$\beta_{(m,e)}(\alpha) = 1 - (\alpha \mathbb{E}[R_{m,B}])\beta_{(m+1,e)}(\alpha)$$

and an application of Leibniz's differentiation formula reveals that for each integer $k \geq 1$,

$$\beta^{(k)}_{(m,e)}(\alpha) = -\left[k\mathbb{E}[R_{m,B}]\beta^{(k-1)}_{(m+1,e)}(\alpha) + \alpha\mathbb{E}[R_{m,B}]\beta^{(k)}_{(m+1,e)}(\alpha)\right]$$

i.e.

$$k\beta^{(k-1)}_{(m+1,e)}(\alpha) + \alpha\beta^{(k)}_{(m+1,e)}(\alpha) + \frac{\beta^{(k)}_{(m,e)}(\alpha)}{\mathbb{E}[R_{m,B}]}.$$

Multiplying both sides by $(-\alpha)^{k-1}/k!$ reveals

$$\frac{(-\alpha)^{k-1}\beta^{(k-1)}_{(m+1,e)}(\alpha)}{(k-1)!} - \frac{(-\alpha)^{k}\beta^{(k)}_{(m+1,e)}(\alpha)}{k!} = \frac{(-\alpha)^k}{k!}\frac{\beta^{(k)}_{(m,e)}(\alpha)}{\alpha\mathbb{E}[R_{m,B}]}$$

or, equivalently,

$$x_{m+1,k-1}(\alpha) - x_{m+1,k}(\alpha) = \frac{x_{m,k}(\alpha)}{\alpha\mathbb{E}[R_{m,B}]}.$$

i.e.

$$x_{m+1,k}(\alpha) = x_{m+1,k-1}(\alpha) - \frac{x_{m,k}(\alpha)}{\alpha\mathbb{E}[R_{m,B}]}.$$

This completes the proof of the claim.

Other recursive schemes can be constructed as well by starting with the formula given in Proposition A.3, then differentiating in various ways. We leave the development of such recursions to the interested reader.

4.3. Conditional distribution of age and residual queueing times of other customers

Another interesting distribution to consider is the amount of time each person currently in the system at time 0 has been in the system. For each $t \in \mathbb{R}$, and each integer $k \geq 2$, let $A_{k,q}(t)$ denote the amount of time the customer present in slot k at time t has spent in the queue by time t, and let $R_{k,q}(t)$ denote the remaining amount of time the customer present in slot k at time t will spend in the system. Notice that for each integer $k \geq 2$, there is no need to keep track of the distribution of the amount of time the customer currently present in slot k at time t will spend in service, as the law of this distribution clearly has CDF F, and is independent of $Q(t)$, $A_{k,q}(t)$, and $R_{k,q}(t)$. Moreover, the joint distribution of $Q(t)$ and $R_{k,q}(t)$ is known once the joint distribution of $Q(t)$ and $R_{1,s}(t)$ is known, but incorporating

$A_{k,q}(t)$ into the joint distribution makes finding the new joint distribution a more complicated task.

It is also possible to derive a reasonably explicit, yet admittedly somewhat unwieldly expression for the joint distribution of the number of customers in the system, and the amount of time the person occupying slot k at time t has been in the system. Our derivation of this joint transform will require us to calculate, for each $t \geq 0$, each integer $n \geq 2$, and each integer $k \in \{2, 3, \ldots, n\}$,

$$\mathbb{E}[e^{-\gamma_1(t-T_{k-1})}\mathbf{1}_{\{N(t)=n-1\}}] = e^{-\gamma_1 t}\mathbb{E}[e^{\gamma_1 T_{k-1}}\mathbf{1}_{\{N(t)=n-1\}}].$$

Proposition 4.2. *For each $t \geq 0$, each integer $n \geq 1$, and each integer $k \in \{1, 2, 3, \ldots, n\}$,*

$$\begin{aligned}\mathbb{E}[e^{\gamma T_k}\mathbf{1}_{\{N(t)=n\}}] =& e^{\gamma t}e^{-\lambda t}\frac{\lambda^n}{\gamma^n}\left[\sum_{\ell=0}^{k-1}(-1)^{\ell}\binom{n-k+\ell}{\ell}\frac{(\gamma t)^{(k-1)-\ell}}{(k-1-\ell)!}\right.\\ &\left.+(-1)^k\sum_{\ell=0}^{n-k}\binom{n-1-\ell}{k-1}\frac{(\gamma t)^{\ell}e^{-\gamma t}}{\ell!}\right]\end{aligned}$$

We will prove this claim with an induction argument: while this is a valid proof, it is not satisfying in terms of understanding where the identity actually comes from. In order to get an idea of where this identity comes from, it helps to write out a few cases, then simplify using Proposition B.1 and Corollary B.1 from the Appendix.

Proof. First consider the case where $k = 1$. For each integer $n \geq 1$,

$$\begin{aligned}\mathbb{E}[e^{\gamma T_1}\mathbf{1}_{\{N(t)=n\}}] &= \int_0^t e^{\gamma s}\frac{(\lambda(t-s))^{n-1}e^{-\lambda(t-s)}}{(n-1)!}\lambda e^{-\lambda s}ds\\ &= e^{-\lambda t}\int_0^t e^{\gamma s}\frac{(\lambda(t-s))^{n-1}}{(n-1)!}ds\\ &= e^{-\lambda t}\int_0^t \frac{(\lambda s)^{n-1}}{(n-1)!}e^{\gamma(t-s)}ds\\ &= e^{-\lambda t}e^{\gamma t}\int_0^t \frac{\lambda(\lambda s)^{n-1}}{(n-1)!}e^{-\gamma s}ds\\ &= e^{-\lambda t}e^{\gamma t}\frac{\lambda^n}{\gamma^n}\left[1-\sum_{\ell=0}^{n-1}\frac{(\gamma t)^{\ell}e^{-\gamma t}}{\ell!}\right].\end{aligned}$$

Proceeding by induction, suppose the result is true for some integer k, for each $n \geq k$. Then for each integer $n \geq k$,

$$\begin{aligned}&\mathbb{E}[e^{\gamma T_{k+1}}\mathbf{1}_{\{N(t)=n+1\}}]\\ &= \int_0^t e^{\gamma s}\mathbb{E}[e^{\gamma T_k}\mathbf{1}_{\{N(t-s)=n\}}]\lambda e^{-\lambda s}ds\end{aligned}$$

$$
\begin{aligned}
&= e^{\gamma t}e^{-\lambda t}\frac{\lambda^{n+1}}{\gamma^{n+1}}\int_0^t \left[\sum_{\ell=0}^{k-1}(-1)^\ell\binom{n-k+\ell}{\ell}\frac{\gamma(\gamma s)^{(k-1)-\ell}}{((k-1)-\ell)!}\right.\\
&\qquad\qquad \left. + (-1)^k\sum_{\ell=0}^{n-k}\binom{n-1-\ell}{k-1}\frac{\gamma(\gamma s)^\ell e^{-\gamma s}}{\ell!}\right] ds\\
&= e^{\gamma t}e^{-\lambda t}\frac{\lambda^{n+1}}{\gamma^{n+1}}\left[\sum_{\ell=0}^{k-1}(-1)^\ell\binom{n-k+\ell}{\ell}\frac{(\gamma t)^{k-\ell}}{(k-\ell)!}\right.\\
&\qquad\qquad \left. + (-1)^k\sum_{\ell=0}^{n-k}\binom{n-1-\ell}{k-1}\left[1-\sum_{j=0}^{\ell}\frac{(\gamma t)^j e^{-\gamma t}}{j!}\right]\right]\\
&= e^{\gamma t}e^{-\lambda t}\frac{\lambda^{n+1}}{\gamma^{n+1}}\left[\sum_{\ell=0}^{k}(-1)^\ell\binom{n-k+\ell}{\ell}\frac{(\gamma t)^{k-\ell}}{(k-\ell)!} + (-1)^{k+1}\sum_{\ell=0}^{n-k}\binom{n-1-\ell}{k-1}\sum_{j=0}^{\ell}\frac{(\gamma t)^j e^{-\gamma t}}{j!}\right]\\
&= e^{\gamma t}e^{-\lambda t}\frac{\lambda^{n+1}}{\gamma^{n+1}}\left[\sum_{\ell=0}^{k}(-1)^\ell\binom{n-k+\ell}{\ell}\frac{(\gamma t)^{k-\ell}}{(k-\ell)!} + (-1)^{k+1}\sum_{j=0}^{n-k}\sum_{\ell=j}^{n-k}\binom{n-1-\ell}{k-1}\frac{(\gamma t)^j e^{-\gamma t}}{j!}\right]\\
&= e^{\gamma t}e^{-\lambda t}\frac{\lambda^{n+1}}{\gamma^{n+1}}\left[\sum_{\ell=0}^{k}(-1)^\ell\binom{n-k+\ell}{\ell}\frac{(\gamma t)^{k-\ell}}{(k-\ell)!} + (-1)^{k+1}\sum_{j=0}^{n-k}\binom{(n+1)-1-j}{(k+1)-1}\frac{(\gamma t)^j e^{-\gamma t}}{j!}\right]
\end{aligned}
$$

which completes the induction argument.

Our next result can be used to determine the joint distribution of $A_{j,q}(0)$ and $R_{j,q}(0)$, conditional on $Q(0) = n$.

Theorem 4.3. *For each integer $n \geq 1$, and each integer $j \in \{2, 3, \ldots, n\}$,*

$$
\begin{aligned}
&\mathbb{E}[e^{-(\gamma_1 A_{j,q}(0)+\gamma_2 R_{j,q}(0))} \mid Q(0) = n]\\
&= \frac{\beta(\gamma_2)^{j-2}}{\pi_n}\frac{\lambda^{n-1}}{\gamma_1^{n-1}}\sum_{\ell=0}^{j-2}(-1)^{j-2-\ell}\binom{n-2-\ell}{n-j}\\
&\quad\times\left[\frac{\gamma_1^\ell}{\lambda^\ell}\sum_{k=0}^{\ell}(\pi_0\mathbf{1}_{\{k=0\}}+\pi_{k+1})\left[\left[\frac{\lambda}{\lambda-\gamma_2}\right]^{\ell-k+1}\beta(\gamma_2) - \sum_{m=0}^{\ell-k}\left[\frac{\lambda}{\lambda-\gamma_2}\right]^{\ell-k+1-m}\frac{(-\lambda)^m}{m!}\beta^{(m)}(\lambda)\right]\right]\\
&\quad+\frac{\beta(\gamma_2)^{j-2}}{\pi_n}\frac{\lambda^{n-1}}{\gamma_1^{n-1}}(-1)^{j-1}\sum_{\ell=0}^{n-j}\binom{n-2-\ell}{j-2}\\
&\quad\times\left[\frac{\gamma_1^\ell}{\lambda^\ell}\sum_{k=0}^{\ell}\frac{\phi_{Q_q}^{(k)}(-\gamma_1/\lambda)}{k!}\left[\left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right]^{\ell-k+1}\beta(\gamma_2)\right.\right.\\
&\qquad\left.\left. - \sum_{m=0}^{\ell-k}\left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right]^{\ell-k+1-m}\frac{(-\lambda)^m}{m!}\beta^{(m)}(\lambda+\gamma_1)\right]\right]
\end{aligned}
$$

Proof. First observe that for each integer $n \geq 1$, and each integer $j \in \{2, 3, \ldots, n\}$,

$$\begin{aligned} &\mathbf{1}_{\{Q(0)=n\}} e^{-(\gamma_1 A_{j,q}(0)+\gamma_2 R_{j,q}(0))} \\ &= \int_{(-\infty,0]} \mathbf{1}_{\{W(s-)\leq -s, W(s-)+B_{-N[-s,0]+1}>-s\}} e^{-\gamma_1(-T_{j-1}(s))} \\ &\quad \times e^{-\gamma_2(W(s-)+B(s)+\int_{(s,T_{j-1}(s))} B_{-N[u,0]+1} du-(-s))} \mathbf{1}_{\{N(s,0]=n-1\}} N(ds). \end{aligned} \tag{21}$$

This observation follows from the fact that $Q(0) = n$ for some integer $n \geq 1$ if and only if there exists a single arrival instant $s < 0$ satisfying the property that the customer who arrived at time s is currently receiving service at time zero (which means $W(s-) \leq -s$ and $W(s-) + B_{-N[-s,0]+1} > -s$) and exactly $n-1$ customers arrived in the interval $(s, 0]$. Furthermore, the customer found in slot j at time zero must have arrived at time

$$T_{j-1}(s) := \inf\{t \geq s : N(s,t] = j-1\}$$

and if we look at the state of the system at time zero, the amount of time that customer has spent in the queue is simply

$$A_{j,q}(0) = 0 - T_{j-1}(s)$$

and the remaining amount of time that customer will spend in the queue is

$$R_{j,q}(0) = W(s-) + B_{-N[-s,0]+1} + \int_{(-s,T_{j-1}(s))} B_{-N[y,0]+1} N(dy) - (-s)$$

as this represents the remaining amount of work present at time zero that is only associated with customers who arrived *before* the customer arriving at time $T_{j-1}(s)$.

After taking the expected value of both sides of (21), while further applying the Campbell-Mecke formula to the right-hand-side, we get

$$\begin{aligned} &\mathbb{E}[\mathbf{1}_{\{Q(0)=n\}} e^{-(\gamma_1 A_{k,q}(0)+\gamma_2 R_{k,q}(0))}] \\ &= \lambda \beta(\gamma_2)^{k-2} \int_{-\infty}^{0} \mathbb{E}[\mathbf{1}_{\{W(0)\leq -s, W(0)+B>-s\}} e^{-\gamma_2(W(0)+B+s)}] \mathbb{E}[e^{-\gamma_1(-T_{k-1}(s))} \mathbf{1}_{\{N(s,0]=n-1\}}] ds \\ &= \lambda \beta(\gamma_2)^{k-2} \int_{0}^{\infty} \mathbb{E}[\mathbf{1}_{\{W(0)\leq s, W(0)+B>s\}} e^{-\gamma_2(W(0)+B-s)}] \mathbb{E}[e^{-\gamma_1(s-T_{k-1})} \mathbf{1}_{\{N(s)=n-1\}}] ds. \end{aligned}$$

Next, observe that

$$\begin{aligned} &\mathbb{E}[e^{-\gamma_1(s-T_{j-1})} \mathbf{1}_{\{N(s)=n-1\}}] \\ &= e^{-\lambda s} \frac{\lambda^{n-1}}{\gamma_1^{n-1}} \left[\sum_{\ell=0}^{j-2} (-1)^{\ell} \binom{n-j+\ell}{\ell} \frac{(\gamma_1 s)^{(j-2)-\ell}}{(j-2-\ell)!} + (-1)^{j-1} \sum_{\ell=0}^{n-j} \binom{n-2-\ell}{j-2} \frac{(\gamma_1 s)^{\ell} e^{-\gamma_1 s}}{\ell!} \right] \end{aligned}$$

which in turn means

$$\mathbb{E}[\mathbf{1}_{\{Q(0)=n\}} e^{-(\gamma_1 A_{j,q}(0)+\gamma_2 R_{j,q}(0))}]$$

$$= \lambda\beta(\gamma_2)^{j-2}\frac{\lambda^{n-1}}{\gamma_1^{n-1}}\sum_{\ell=0}^{j-2}(-1)^{\ell}\binom{n-j+\ell}{\ell}\mathbb{E}\left[\int_{W(0)}^{W(0)+B} e^{-\gamma_2(W(0)+B-s)}\frac{(\gamma_1 s)^{(j-2)-\ell}}{(j-2-\ell)!}e^{-\lambda s}ds\right]$$
$$+ \lambda\beta(\gamma_2)^{j-2}\frac{\lambda^{n-1}}{\gamma_1^{n-1}}(-1)^{j-1}\sum_{\ell=0}^{n-j}\binom{n-2-\ell}{j-2}\mathbb{E}\left[\int_{W(0)}^{W(0)+B} e^{-\gamma_2(W(0)+B-s)}\frac{(\gamma_1 s)^{\ell}e^{-\gamma_1 s}}{\ell!}e^{-\lambda s}ds\right]$$
$$= \lambda\beta(\gamma_2)^{j-2}\frac{\lambda^{n-1}}{\gamma_1^{n-1}}\sum_{\ell=0}^{j-2}(-1)^{\ell}\binom{n-j+\ell}{\ell}\mathbb{E}\left[\int_{0}^{B} e^{-\gamma_2(B-s)}\frac{(\gamma_1 (s+W(0)))^{(j-2)-\ell}}{(j-2-\ell)!}e^{-\lambda(s+W(0))}ds\right]$$
$$+ \lambda\beta(\gamma_2)^{j-2}\frac{\lambda^{n-1}}{\gamma_1^{n-1}}(-1)^{j-1}\sum_{\ell=0}^{n-j}\binom{n-2-\ell}{j-2}\mathbb{E}\left[\int_{0}^{B} e^{-\gamma_2(B-s)}\frac{(\gamma_1 (s+W(0)))^{\ell}}{\ell!}e^{-(\lambda+\gamma_1)(s+W(0))}ds\right]$$
$$= \lambda\beta(\gamma_2)^{j-2}\frac{\lambda^{n-1}}{\gamma_1^{n-1}}\sum_{\ell=0}^{j-2}(-1)^{j-2-\ell}\binom{n-2-\ell}{n-j}\mathbb{E}\left[\int_{0}^{B} e^{-\gamma_2(B-s)}\frac{(\gamma_1 (s+W(0)))^{\ell}}{\ell!}e^{-\lambda(s+W(0))}ds\right]$$
$$+ \lambda\beta(\gamma_2)^{j-2}\frac{\lambda^{n-1}}{\gamma_1^{n-1}}(-1)^{j-1}\sum_{\ell=0}^{n-j}\binom{n-2-\ell}{j-2}\mathbb{E}\left[\int_{0}^{B} e^{-\gamma_2(B-s)}\frac{(\gamma_1 (s+W(0)))^{\ell}}{\ell!}e^{-(\lambda+\gamma_1)(s+W(0))}ds\right]$$

Furthermore, since

$$\lambda\mathbb{E}\left[\int_{0}^{B} e^{-\gamma_2(B-s)}\frac{(\gamma_1 (s+W(0)))^{\ell}}{\ell!}e^{-\lambda(s+W(0))}ds\right]$$
$$= \lambda\sum_{k=0}^{\ell}\mathbb{E}\left[\frac{(\gamma_1 W(0))^k}{k!}e^{-\lambda W(0)}\right]\mathbb{E}\left[\int_0^B e^{-\gamma_2(B-s)}\frac{(\gamma_1 s)^{\ell-k}}{(\ell-k)!}e^{-\lambda s}ds\right]$$
$$= \frac{\gamma_1^{\ell}}{\lambda^{\ell}}\sum_{k=0}^{\ell}(\pi_0\mathbf{1}_{\{k=0\}}+\pi_{k+1})\mathbb{E}\left[\int_0^B e^{-\gamma_2(B-s)}\frac{\lambda(\lambda s)^{\ell-k}}{(\ell-k)!}e^{-\lambda s}ds\right]$$
$$= \frac{\gamma_1^{\ell}}{\lambda^{\ell}}\sum_{k=0}^{\ell}(\pi_0\mathbf{1}_{\{k=0\}}+\pi_{k+1})\left[\left[\frac{\lambda}{\lambda-\gamma_2}\right]^{\ell-k+1}\beta(\gamma_2)-\sum_{m=0}^{\ell-k}\left[\frac{\lambda}{\lambda-\gamma_2}\right]^{\ell-k+1-m}\frac{(-\lambda)^m}{m!}\beta^{(m)}(\lambda)\right]$$

and likewise,

$$\lambda\mathbb{E}\left[\int_{0}^{B} e^{-\gamma_2(B-s)}\frac{(\gamma_1 (s+W(0)))^{\ell}}{\ell!}e^{-(\lambda+\gamma_1)(s+W(0))}ds\right]$$
$$= \lambda\sum_{k=0}^{\ell}\mathbb{E}\left[\frac{(\gamma_1 W(0))^k}{k!}e^{-(\lambda+\gamma_1)W(0)}\right]\mathbb{E}\left[\int_0^B e^{-\gamma_1 s}e^{-\gamma_2(B-s)}\frac{(\gamma_1 s)^{\ell-k}}{(\ell-k)!}e^{-\lambda s}ds\right]$$
$$= \frac{\gamma_1^{\ell}}{\lambda^{\ell}}\sum_{k=0}^{\ell}\frac{\phi_{Q_q}^{(k)}(-\gamma_1/\lambda)}{k!}\mathbb{E}\left[\int_0^B e^{-\gamma_1 s}e^{-\gamma_2(B-s)}\frac{\lambda(\lambda s)^{\ell-k}}{(\ell-k)!}e^{-\lambda s}ds\right]$$
$$= \frac{\gamma_1^{\ell}}{\lambda^{\ell}}\sum_{k=0}^{\ell}\frac{\phi_{Q_q}^{(k)}(-\gamma_1/\lambda)}{k!}\left[\left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2}\right]^{\ell-k+1}\beta(\gamma_2)\right.$$

$$- \sum_{m=0}^{\ell-k} \left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2} \right]^{\ell-k+1-m} \frac{(-\lambda)^m}{m!} \beta^{(m)}(\lambda+\gamma_1) \Bigg]$$

we conclude that

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}_{\{Q(0)=n\}} e^{-(\gamma_1 A_{j,q}(0)+\gamma_2 R_{j,q}(0))}] \\
&= \beta(\gamma_2)^{j-2} \frac{\lambda^{n-1}}{\gamma_1^{n-1}} \sum_{\ell=0}^{j-2} (-1)^{j-2-\ell} \binom{n-2-\ell}{n-j} \\
&\times \left[\frac{\gamma_1^\ell}{\lambda^\ell} \sum_{k=0}^{\ell} (\pi_0 \mathbf{1}_{\{k=0\}} + \pi_{k+1}) \left[\left[\frac{\lambda}{\lambda-\gamma_2} \right]^{\ell-k+1} \beta(\gamma_2) - \sum_{m=0}^{\ell-k} \left[\frac{\lambda}{\lambda-\gamma_2} \right]^{\ell-k+1-m} \frac{(-\lambda)^m}{m!} \beta^{(m)}(\lambda) \right] \right] \\
&+ \beta(\gamma_2)^{j-2} \frac{\lambda^{n-1}}{\gamma_1^{n-1}} (-1)^{j-1} \sum_{\ell=0}^{n-j} \binom{n-2-\ell}{j-2} \\
&\times \left[\frac{\gamma_1^\ell}{\lambda^\ell} \sum_{k=0}^{\ell} \frac{\phi_{Q_q}^{(k)}(-\gamma_1/\lambda)}{k!} \left[\left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2} \right]^{\ell-k+1} \beta(\gamma_2) \right. \right. \\
&\qquad \left. \left. - \sum_{m=0}^{\ell-k} \left[\frac{\lambda}{\lambda+\gamma_1-\gamma_2} \right]^{\ell-k+1-m} \frac{(-\lambda)^m}{m!} \beta^{(m)}(\lambda+\gamma_1) \right] \right]
\end{aligned}$$

from which we get the result.

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Appendix

This Appendix contains a collection of results that are used to establish the main results of this study.

A. Results on Residual Distirbutions

Given a nonnegative random variable B having CDF F and LST β that also satisfies $\mathbb{E}[B] < \infty$, we define the nonnegative random variable R_B as a nonnegative random variable whose CDF $F_{(1,e)}$ is defined as follows: for each $t \geq 0$,

$$F_{(1,e)}(t) := \mathbb{P}(R_B \leq t) = \int_0^t \frac{\mathbb{P}(B > x)}{\mathbb{E}[B]} dx$$

which is a well-defined, proper CDF when $\mathbb{E}[B] < \infty$. It is very well-known (and easy to show) that the LST of R_B, which we represent as $\beta_{(1,e)}$, is simply

$$\beta_{(1,e)}(\alpha) = \frac{1 - \beta(\alpha)}{\alpha \mathbb{E}[B]}.$$

We can also define 'residuals of residuals', in that for each integer $m \geq 1$, when $\mathbb{E}[B^{m+1}] < \infty$, we can define the random variable $R_{m+1,B}$ as a nonnegative random variable having

CDF $F_{(m+1,e)}$, given by

$$F_{(m+1,e)}(t) := \int_0^t \frac{\mathbb{P}(R_{m,B} > x)}{\mathbb{E}[R_{m,B}]} dx$$

which, again, is a well-defined, proper CDF when $\mathbb{E}[B^{m+1}] < \infty$.

It is convenient to refer to $R_{m,B}$ as the m-th order residual of B, where the follow the convention that $R_{0,B}$ is equal in distribution to B. Such higher-order residuals have been considered previously: a recent example of work involving these distributions includes that of Kerner and Löpker [19], and other work featuring these higher-order residuals can be found in the references cited therein.

Our first result is well-known (see e.g. [19]) and provides an expression for the moments of $R_{n,B}$, when they exist.

Proposition A.1. *For each integer $n \geq 0$, and each integer $k \geq 0$,*

$$\mathbb{E}[R_{n,B}^k] = \frac{\mathbb{E}[B^{k+n}]}{\binom{k+n}{n}\mathbb{E}[B^n]}.$$

Proof. First, observe that for each integer $n \geq 1$, and each integer $k \geq 1$,

$$\begin{aligned}\mathbb{E}[R_{n,B}^k] = \int_0^\infty x^k dF_{(n,e)}(x) &= \frac{1}{\mathbb{E}[R_{n-1,B}]} \int_0^\infty x^k \mathbb{P}(R_{n-1,B} > x) dx \\ &= \frac{1}{(k+1)\mathbb{E}[R_{n-1,B}]} \int_0^\infty (k+1) x^k \mathbb{P}(R_{n-1,B} > x) dx \\ &= \frac{\mathbb{E}[R_{n-1,B}^{k+1}]}{(k+1)\mathbb{E}[R_{n-1,B}]}.\end{aligned}$$

Further iterations of this identity yield

$$\mathbb{E}[R_{n,B}^k] = \frac{\mathbb{E}[R_{n-2,B}^{k+2}]}{(k+1)(k+2)\mathbb{E}[R_{n-1,B}]\mathbb{E}[R_{n-2,B}]} = \cdots = \frac{\mathbb{E}[B^{k+n}]}{\left[\prod_{\ell=1}^n (k+\ell)\right] \prod_{\ell=0}^{n-1} \mathbb{E}[R_{\ell,B}]}.$$

The same line of reasoning reveals that

$$\mathbb{E}[R_{n-1,B}] = \frac{\mathbb{E}[B^n]}{\left[\prod_{\ell=1}^{n-1}(1+\ell)\right] \prod_{\ell=0}^{n-2} \mathbb{E}[R_{\ell,B}]} = \frac{\mathbb{E}[B^n]}{n! \prod_{\ell=0}^{n-2} \mathbb{E}[R_{\ell,B}]}$$

which implies

$$\prod_{\ell=0}^{n-1} \mathbb{E}[R_{\ell,B}] = \frac{\mathbb{E}[B^n]}{n!}$$

so in conclusion,

$$\mathbb{E}[R_{n,B}^k] = \frac{\mathbb{E}[B^{k+n}]}{\left[\prod_{\ell=1}^n (k+\ell)\right](1/(n!))\mathbb{E}[B^n]} = \frac{\mathbb{E}[B^{k+n}]}{\binom{k+n}{n}\mathbb{E}[B^n]}$$

proving the claim.

Proposition A.2. *For each integer $n \geq 0$,*

$$\mathbb{P}(R_{n,B} > x) = \frac{\mathbb{E}[(B-x)^n \mathbf{1}_{\{B>x\}}]}{\mathbb{E}[B^n]}$$

for each $x > 0$.

Proof. We give the proof here: it can also be found in [14], but while the result the correct, there is one misleading step in the proof found in [14], which we correct here.

The result is trivially true for the case where $n = 0$, and furthermore,

$$\begin{aligned}
\mathbb{E}[(B-x)\mathbf{1}_{\{B>x\}}] &= \int_x^\infty (y-x)dF(y) \\
&= \int_x^\infty \int_x^y dzdF(y) \\
&= \int_x^\infty \int_z^\infty dF(y)dz \\
&= \int_x^\infty \mathbb{P}(B > z)dx = \mathbb{E}[B]\mathbb{P}(R_{1,B} > x)
\end{aligned}$$

which proves the claim for the case where $n = 1$. Proceeding by strong induction, we get

$$\begin{aligned}
\mathbb{E}[(B-x)^{n+1}\mathbf{1}_{\{B>x\}}] &= \int_x^\infty (y-x)^{n+1}dF(y) \\
&= \int_x^\infty \int_x^y (y-x)^n dudF(y) \\
&= \int_x^\infty \int_u^\infty (y-x)^n dF(y)du \\
&= \sum_{k=0}^n \binom{n}{k} \int_x^\infty (u-x)^{n-k} \int_u^\infty (y-u)^k dF(y)du \\
&= \sum_{k=0}^n \binom{n}{k} \mathbb{E}[B^k] \int_x^\infty (u-x)^{n-k} \mathbb{P}(R_{k,B} > u)du.
\end{aligned}$$

Next, observe that for $1 \leq k \leq n$,

$$\begin{aligned}
\binom{n}{k}\mathbb{E}[B^k]\int_x^\infty (u-x)^{n-k}\mathbb{P}(R_{k,B} > u)du &= \frac{1}{n+1-k}\mathbb{E}[(R_{k,B}-x)^{n-k+1}\mathbf{1}_{\{R_{k,B}>x\}}] \\
&= \frac{1}{n+1-k}\binom{n}{k}\mathbb{E}[B^k]\mathbb{E}[R_{k,B}^{n-k+1}]\mathbb{P}(R_{n+1,B} > x)
\end{aligned}$$

and

$$\binom{n}{k}\mathbb{E}[B^k]\mathbb{E}[R_{k,B}^{n-k+1}] = \frac{1}{n+1-k}\binom{n}{k}\mathbb{E}[B^k]\frac{\mathbb{E}[B^{n+1}]}{\binom{n+1}{k}\mathbb{E}[B^k]}$$

$$= \frac{1}{(n+1-k)} \frac{n!}{k!(n-k)!} \frac{k!(n+1-k)!}{(n+1)!} \mathbb{E}[B^{n+1}] = \frac{\mathbb{E}[B^{n+1}]}{n+1}$$

which in turn implies

$$\mathbb{E}[(B-x)^{n+1}\mathbf{1}_{\{B>x\}}] = \frac{1}{n+1}\mathbb{E}[(B-x)^{n+1}\mathbf{1}_{\{B>x\}}] + \sum_{k=1}^{n} \frac{\mathbb{E}[B^{n+1}]}{n+1}\mathbb{P}(R_{n+1,B} > x)$$

from which we arrive at the claim.

Many of our results will be stated in terms of higher-order derivatives of Laplace transforms of higher-order residuals of the service time distribution. The next known result provides us with a simple way to evaluate these derivatives.

Proposition A.3. *Suppose* $\mathbb{E}[B^n] < \infty$. *Then*

$$\beta(\alpha) = \sum_{k=0}^{n-1} (-\alpha)^k \frac{\mathbb{E}[B^k]}{k!} + \frac{(-\alpha)^n \mathbb{E}[B^n]}{n!} \beta_{(n,e)}(\alpha).$$

Proof. This can be established simply by repeated iterations of the following formula: for each integer $n \geq 0$,

$$\beta_{(n,e)}(\alpha) = 1 - (\alpha \mathbb{E}[R_{n,B}])\beta_{(n+1,e)}(\alpha)$$

where $\beta_{(0,e)}(\alpha) = \beta(\alpha)$.

Proposition A.4. *Let* B *be a nonnegative random variable. Then for each integer* $n \geq 0$,

$$\int_0^\infty e^{-\gamma s}\mathbb{P}(B > s)\frac{(\lambda s)^n e^{-\lambda s}}{n!} ds = \mathbb{E}[B]\frac{(-\lambda)^n \beta_{(1,e)}^{(n)}(\lambda + \gamma)}{n!}.$$

Proof. Here

$$\begin{aligned}\int_0^\infty e^{-\gamma s}\mathbb{P}(B > s)\frac{(\lambda s)^n e^{-\lambda s}}{n!} ds &= \lambda^n \mathbb{E}[B] \int_0^\infty \frac{s^n e^{-(\lambda+\gamma)s}}{n!} \frac{\mathbb{P}(B > s)}{\mathbb{E}[B]} ds \\ &= \mathbb{E}[B]\frac{(-\lambda)^n \beta_{(1,e)}^{(n)}(\lambda + \gamma)}{n!}\end{aligned}$$

proving the claim.

B. A Useful Summation Identity

The following proposition is most likely well-known to experts in discrete mathematics and theoretical computer science, but we were not aware of it previously, so in order to help the reader we will provide a clear statement of the result, as well as a proof.

Proposition B.1. *Let $\{a_\ell\}_{\ell\geq 0}$ be any sequence of complex numbers. For each integer $n \geq 0$, and each integer $k \geq 1$, we have*

$$\sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{\ell_1}\cdots\sum_{\ell_k=0}^{\ell_{k-1}} a_{\ell_k} = \sum_{j=0}^{n}\binom{n+(k-1)-j}{k-1}a_j.$$

Proof. It takes a little experimentation to actually see the pattern, but once the pattern has been found, an induction proof (induction on k) is easy to give. First, for each integer $n \geq 0$,

$$\sum_{\ell_1=0}^{n} a_{\ell_1} = \sum_{j=0}^{n} a_j = \sum_{j=0}^{n}\binom{n+(1-1)-j}{0}a_j$$

so the statement is trivially true for all $n \geq 0$ when $k = 1$.

Assume now that the statement is true for a fixed k (and for each $n \geq 0$). Then for each integer $n \geq 0$,

$$\begin{aligned}
\sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{\ell_1}\cdots\sum_{\ell_k=0}^{\ell_{k-1}}\sum_{\ell_{k+1}=0}^{\ell_k} a_{\ell_{k+1}} &= \sum_{\ell_1=0}^{n}\left[\sum_{\ell_2=0}^{\ell_1}\cdots\sum_{\ell_k=0}^{\ell_{k-1}}\sum_{\ell_{k+1}=0}^{\ell_k} a_{\ell_{k+1}}\right] \\
&= \sum_{\ell_1=0}^{n}\sum_{j=0}^{\ell_1}\binom{\ell_1+(k-1)-j}{k-1}a_j \\
&= \sum_{j=0}^{n}\sum_{\ell_1=j}^{n}\binom{\ell_1+(k-1)-j}{k-1}a_j \\
&= \sum_{j=0}^{n}\binom{n+(k+1-1)-j}{k+1-1}a_j
\end{aligned}$$

which proves the claim.

The next corollary follows from an application of Proposition B.1, combined with what is often referred to as the 'hockey-stick identity': for each integer $k \geq 0$, and each integer $n \geq 0$,

$$\sum_{\ell=0}^{n}\binom{k+\ell}{k} = \binom{k+n+1}{k+1}.$$

Corollary B.1. *For each integer $n \geq 0$, and each integer $k \geq 1$, we have*

$$\sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{\ell_1}\cdots\sum_{\ell_k=0}^{\ell_{k-1}}(1) = \binom{n+k}{k}.$$

Proof. Using Proposition B.1, combined with the hockey-stick identity, we get

$$\sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{\ell_1}\cdots\sum_{\ell_k=0}^{\ell_{k-1}}(1) = \sum_{j=0}^{n}\binom{n+(k-1)-j}{k-1} = \binom{n+k}{k}$$

proving the claim.