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$MAP/E_k/1$ Queue with Working Vacation Providing Main Service Only in Normal Mode of Service

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Abstract: In this paper, we consider a $MAP/E_k/1$ queue with working vacation. Customers arrive according to a Markovian Arrival Process and service time follows generalized Erlang distribution of order n. Service in the first k stages is called the preliminary service and service in the remaining $n-k$ stages is called the main service. When the system becomes empty at the time of completion of a service, the server goes on working vacation. During working vacation server provides only the preliminary service. After availing of the preliminary service, the customers leave the system with probability p. Those who require the main service, join a buffer of finite capacity N with complementary probability $1-p$. The server switches to normal mode when the vacation expires, or N customers accumulate in the buffer during working vacation, whichever occurs first. The customer in service at the working vacation expiration epoch, continues to get his service in normal mode. Steady state analysis of this system is performed. Several performance characteristics of interest are computed. A cost function is constructed and the optimal values of N for the positive, zero and negative correlation values of the Markovian arrival process are obtained.

Keywords: Erlang distribution, main service, Markovian arrival process (MAP), N-policy, preliminary service, working vacation.

1. Introduction

The queueing system with server vacations has been well-studied since the late 1970's. Considering the importance of the subject, several researchers have been attracted to it, and a good amount of studies have been conducted, especially from the early 1980's. The first review paper on vacation queueing models is by Doshi [2]. Several researchers concentrated

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on the classical queueing system and extended to vacation queueing models by allowing idle servers to work on non-queueing jobs. Hence the vacation models are more applicable in several areas, especially in the flexible manufacturing, computer and communications systems, etc. Because of the acceptability of these applications, more researchers have conducted studies on vacation models during the late 1980's and the entire of the decade 1990. These were surveyed in the book of Takagi [11] in 1991 and Tian and Zhang [12] in 2006.

Vacation in the queueing system takes place either because of the absence of customers at a service completion epoch or due to server breakdown. The advantage of this server vacation system is that it can utilize its time for other purposes. So it makes the queueing model applicable to various real-world service systems. In the vacation queuing system, the server does not provide service when he is on vacation. In contrast, in the working vacation scheme, the server works at a different rate instead of remaining idle/allotted some other work during the vacation period. A queueing system with server vacation was first discussed in the paper by Levy and Yechiali [5]. Considering the scope of wide applications in computer systems, communication networks, production management, etc., extensive studies have been conducted in Markovian queueing systems with working vacations. Servy and Finn [7] introduced the concept of working vacation in which the server offers service at a low rate during vacation if customers are available.

The concept of the N-policy was introduced by Yadin and Naor [14]. It means the server provides service only when N customers accumulate in the systems on completion of a busy period. Extensive studies on vacation queueing systems under N-policy have been conducted since 1963. The N-policy makes the queueing model more applicable in various scenarios, especially optimal management policy, computer processing, manufacturing, transportation systems and so on.

In addition to Ke et al. [3], Tian et al. [13] and Panta et al. [6] are the review papers on vacation queueing model. A review paper by Chandrasekaran et al. [1] provide the latest research results on working vacation queueing systems.

Sreenivasan et al. [10] consider a working vacation queueing system in which the server goes on vacation when the system becomes empty. On return the server provides service at a low rate to customers joining the system. The vacation terminates when either the number of customers in the system reaches N or the vacation clock realises. Krishnamoorthy et al. [4] consider two single server queueing models with non-preemptive priority and working vacation under two distinct N-policies. Sinulal et al. [9] analyse a queueing system in which the service is provided at two stations, station 1 and station 2, operating in tandem. Station 1 is a multi-server station with c identical servers working in parallel, and station 2 is equipped with a single server called the specialist server. An arriving customer enters directly into service at station 1 if at least one of the servers is idle, otherwise he joins an infinite capacity queue. After receiving service at station 1, customers proceed either to station 2 or can exit the system. There is a finite buffer between the two stations. When the buffer is not full, a customer coming out of station 1 joins the buffer with probability p or leaves the system with the complementary probability $1 - p$. The server at station 2 will be turned on only if the number of customers in the buffer reaches a threshold.

In the paper [8], authors consider a single server queueing system with a working vacation. Service has n stages. Service in the first k stages is called the preliminary service and service in the remaining $n-k$ stages is called the main service. When the system becomes empty at the time of completion of service, the server goes on working vacation. Customers who arrive during working vacation are provided only the main service. The server switches to normal mode when the vacation expires, or N customers are served during a working vacation, whichever occurs first. The customer in service at the working vacation expiration epoch is served from the very beginning.

In the present model, we consider a single server queueing system in which customers arrive according to Markovian Arrival Process. Service time follows generalized Erlang distribution of order n. Service in the first k stages is called the preliminary service and those in the remaining $n-k$ stages is called the main service. When the system becomes empty at the time of completion of a service, the server goes on a working vacation. During working vacation the server provides only the preliminary service. After availing of the preliminary service, the customers leave the system with probability p. Those who require the main service to join a buffer of finite capacity N with complementary probability $1-p$. The server switches to normal mode when the vacation expires or N customers accumulate in the buffer during working vacation, whichever occurs first. The customer in service at the working vacation expiration epoch will receive his service in normal mode.

We provide a few real-life examples which illustrate the queueing model described in this paper. Suppose we are going to a tourist place where the tour operators are conducting boat trips for sightseeing. An entrance ticket is issued anytime during working hours from the first counter. However, to get the tickets for the boat ride, the tourists have to wait until there is a specified minimum number of passengers for a new trip. During busy hours, the visitors may not have to spend long time in the waiting area as there are many visitors. Nevertheless, during slack hours, tourists must wait for the boat ride. Another example is hospitals where the Outpatients can get OP tickets during the entire OP hours. The initial medical examinations, such as blood pressure, weight, pulse, etc., are recorded in the screening room. Then they wait for consultation. Doctors conduct inpatient ward visits or other duties during OP hours if no patient is waiting for consultation. But as the number of patients in the OP queue reaches a specific number, the doctor returns to continue the OP consultation.

Salient features of the model discussed in this paper are

- The n service stages are divided into two parts.
- In the working vacation mode, the server provides only the preliminary service.
- In the above mode, after availing of preliminary service, the customer can either leave the system or he can join a buffer of finite capacity.
- Vacation is realized only when the vacation clock expires, or N customers accumulate in the buffer, whichever occurs first.

Notations and abbreviations used in this paper are

- CTMC: Continuous time Markov chain.
- I_a: Identity matrix of order a.

- LIQBD: Level independent Quasi-Birth and Death.
- MAP: Markovian Arrival Process.
- OP: Outpatient.
- $\boldsymbol{e}$: Column vector of $1'$s of appropriate order.
- $\boldsymbol{e}_c(d)$: Column vector of order d with 1 in the c^{th} position and the remaining entries are zero.
- $\bar{\boldsymbol{e}}_b(a)$: Row vector of order a with 1 in the b^{th} position and the remaining entries are zero.

The remaining part of this paper is arranged as follows. In Section 2, the model under study is mathematically formulated. In Section 3, we perform the steady-state analysis of the queueing model under study. The waiting time analysis of a tagged customer is provided in Section 4. Some additional performance measures are computed and presented in Section 5. A cost function is constructed to find the optimal N in Section 6. Numerical results are discussed in Section 7. Finally, in Section 8, optimal N values are found for MAP with positive correlation, zero correlation and negative correlation.

2. Mathematical formulation

We consider a single server queueing system in which customers arrive according to a Markovian Arrival Process with representation (D_0, D_1) of order m. Let δ be an invariant vector of $D = D_0 + D_1$ that is, $\delta D = 0, \delta e = 1$. Service time follows the generalised Erlang distribution of order n. Service in the first k stages is called the preliminary service; service time in each of these stages is exponentially distributed with parameter θ. Service in the remaining $n-k$ stages is called main service and service time in each of these stages is exponentially distributed with parameter ϕ. When the system becomes empty at the time of completion of service, the server goes on a working vacation. During working vacation server provides only the preliminary service. After availing preliminary service, a customer leaves the system with probability p; those who require the main service also join a buffer of finite capacity N (with probability $1-p$). The duration of working vacation is exponentially distributed with parameter η. The server switches to normal mode when the vacation expires or N customers accumulated in the buffer during working vacation, whichever occurs first. The customer in service during the working vacation expiration epoch continues to receive his service in normal mode. Once the working vacation is over, the server start serving customers in the buffer in the order in which they entered it before proceeding to serve those waiting in the main queue.

2.1. The QBD process

The model described in section 1 can be studied as an LIQBD process. First, we define the following notations:
At time t,

$\mathcal{N}(t)$: Number of customers in the queue ,

$$J(t) = \begin{cases} 0, & \text{if the server is in vacation mode.} \\ 1, & \text{if the server is in normal mode.} \end{cases}$$

$M(t)$: Number of customers in the buffer,
$S(t)$: The phase of service,
$A(t)$: The phase of arrival.

$\{(\mathcal{N}(t), M(t), J(t), S(t), A(t)) : t \geq 0\}$ is an LIQBD process with state space
$\Omega = \{\{(0,0,0,*,j) : 1 \leq j \leq m\} \bigcup \{(0,h,0,*,j) : 1 \leq h \leq (N-1); 1 \leq j \leq m\} \bigcup \{(q,h,0,i,j) : q \geq 0; 0 \leq h \leq (N-1); 1 \leq i \leq k; 1 \leq j \leq m\} \bigcup \{(q,h,1,i,j) : q \geq 0; 0 \leq h \leq (N-1); 1 \leq i \leq n; 1 \leq j \leq m\}\}$.
In the absence of customers in the system, no service can be provided; this is indicated by '*' in the position of service coordinate (fourth coordinate in the 5-tuple).

The infinitesimal generator of this CTMC is

$$\mathcal{Q}^* = \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_0 & & & \\ \mathcal{B}_2 & \mathcal{A}_1 & \mathcal{A}_0 & & \\ & \mathcal{A}_2 & \mathcal{A}_1 & \mathcal{A}_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Here $\mathcal{B}_1$ is a square matrix of order $Nm(k+n)+Nm$ which contains the transition rates within the level 0; $\mathcal{B}_0$ is a $(Nm(k+n)+Nm) \times Nm(k+n)$ matrix which contains transition rates from level 0 to level 1; $\mathcal{B}_2$ is a $Nm(k+n) \times (Nm(k+n)+Nm)$ matrix which contains transition rates from level 1 to level 0; $\mathcal{A}_0$ represents transition rates from n to $n+1$ for $n \geq 1$, $\mathcal{A}_1$ represents transition rates within n for $n \geq 1$ and $\mathcal{A}_2$ represents transition rates from n to $n-1$ for $n \geq 2$. All these are square matrices of order $Nm(k+n)$.

$$\mathcal{B}_1 = \begin{bmatrix} D_0 & C_8 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ P & C_0 & C_1 & Q & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ R & \mathbf{0} & C_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_3 & C_4 & C_8 & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & P & C_0 & C_1 & Q & \mathbf{0} & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_5 & \mathbf{0} & \mathbf{0} & C_2 & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & C_3 & C_4 & C_8 & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & & & & & \ddots & \ddots & \ddots & \ddots & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & C_3 & C_4 & C_8 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & P & C_0 & C_6 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & C_5 & \mathbf{0} & \mathbf{0} & C_2 \end{bmatrix}.$$

$$\mathcal{B}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} \\ I_k \otimes D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_n \otimes D_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_k \otimes D_1 & \mathbf{0} & \mathbf{0} & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_n \otimes D_1 & \mathbf{0} & & \mathbf{0} & \mathbf{0} \\ & & & \ddots & \ddots & \ddots & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & I_k \otimes D_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & I_n \otimes D_1 \end{bmatrix}.$$

$$\mathcal{B}_2 = \begin{bmatrix} \mathbf{0} & S & \mathbf{0} & \mathbf{0} & T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_7 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & S & \mathbf{0} & \mathbf{0} & T & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & S & \mathbf{0} & \mathbf{0} & T & & \mathbf{0} & \mathbf{0} \\ & & & & & & & \ddots & \ddots & \ddots & \ddots & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & S & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} & \mathbf{0} \end{bmatrix}, \text{ where}$$

$$C_0 = \begin{bmatrix} D_0 - I_m\theta - I_m\eta & I_m\theta & & & \\ & D_0 - I_m\theta - I_m\eta & I_m\theta & & \\ & \ddots & \ddots & & \\ & & & D_0 - I_m\theta - I_m\eta & I_m\theta \\ & & & & D_0 - I_m\eta - I_m\theta \end{bmatrix}$$

is a square matrix of order mk.

$C_1 = \begin{bmatrix} \eta I_{km} & \mathbf{0} \end{bmatrix}$ is a $mk \times mn$ matrix.

$$C_2 = \begin{bmatrix} D_0 - I_m\theta & I_m\theta & & & & & & \\ & D_0 - I_m\theta & I_m\theta & & & & & \\ & \ddots & \ddots & & & & & \\ & & D_0 - I_m\theta & I_m\theta & & & & \\ & & & D_0 - I_m\phi & I_m\phi & & & \\ & & & \ddots & \ddots & & & \\ & & & & & D_0 - I_m\phi & I_m\phi \\ & & & & & & D_0 - I_m\phi \end{bmatrix}$$

is a square matrix of order mn.

$C_3 = \begin{bmatrix} \bar{\boldsymbol{e}}_{k+1}(n) \otimes \eta I_m \end{bmatrix}$ is a $m \times mn$ matrix.

$C_4 = \begin{bmatrix} D_0 - I_m\eta \end{bmatrix}$ is a $m \times m$ matrix.

$C_5 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \bar{\boldsymbol{e}}_{k+1}(k+1) \otimes I_m\phi & \mathbf{0} \end{bmatrix}$ is a square matrix of order mn.

$C_6 = \begin{bmatrix} I_k \otimes \eta I_m & \boldsymbol{e}_k(k) \otimes \theta(1-p)I_m & \mathbf{0} \end{bmatrix}$ is a $mk \times mn$ matrix.

$C_7 = \begin{bmatrix} \boldsymbol{e}_n(n) \otimes \phi I_m & \mathbf{0} \end{bmatrix}$ is a square matrix of order mn.

$C_8 = \begin{bmatrix} D_1 & \mathbf{0} \end{bmatrix}$ is a $m \times mk$ matrix.

$P = \begin{bmatrix} \boldsymbol{e}_k(k) \otimes \theta p I_m \end{bmatrix}$ is a $mk \times m$ matrix.

$Q = \begin{bmatrix} \boldsymbol{e}_k(k) \otimes \theta(1-p) I_m \end{bmatrix}$ is a $mk \times m$ matrix.

$R = \begin{bmatrix} \boldsymbol{e}_n(n) \otimes \phi I_m \end{bmatrix}$ is a $mn \times m$ matrix.

$S = \begin{bmatrix} \boldsymbol{e}_k(k) \otimes \theta p I_m & \mathbf{0} \end{bmatrix}$ is a square matrix of order mk.

$T = \begin{bmatrix} \boldsymbol{e}_k(k) \otimes \theta(1-p) I_m & \mathbf{0} \end{bmatrix}$ is a square matrix of order mk .

Let $E_1 = \begin{bmatrix} C_0 & C_1 \\ \mathbf{0} & C_2 \end{bmatrix}$; $E_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_5 \end{bmatrix}$; $E_3 = \begin{bmatrix} C_0 & C_6 \\ \mathbf{0} & C_2 \end{bmatrix}$;
$E_4 = \begin{bmatrix} I_k \otimes D_1 & \mathbf{0} \\ \mathbf{0} & I_n \otimes D_1 \end{bmatrix}$; $E_5 = \begin{bmatrix} S & \mathbf{0} \\ \mathbf{0} & C_7 \end{bmatrix}$; $E_6 = \begin{bmatrix} T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$; $E_7 = \begin{bmatrix} S & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

$\mathcal{A}_0 = I_N \otimes E_4$

$$\mathcal{A}_2 = \begin{bmatrix} E_5 & E_6 & & & & & \\ & E_7 & E_6 & & & & \\ & & E_7 & E_6 & & & \\ & & & \ddots & \ddots & & \\ & & & & E_7 & E_6 & \\ & & & & & E_7 & E_6 \\ & & & & & & E_7 \end{bmatrix}.$$

$$\mathcal{A}_1 = \begin{bmatrix} E_1 & \mathbf{0} & & & & & \\ E_2 & E_1 & & & & & \\ & E_2 & E_1 & & & & \\ & & \ddots & \ddots & & & \\ & & & E_2 & E_1 & & \\ & & & & E_2 & E_1 & \\ & & & & & E_2 & E_3 \end{bmatrix}.$$

3. Steady State Analysis

In this section, we perform the steady-state analysis of the queueing model under study. The generator matrix is $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2$.

$$\mathcal{A} = \begin{bmatrix} E_1+E_4+E_5 & E_6 & \mathbf{0} & & & \\ E_2 & E_1+E_4+E_7 & E_6 & & & \\ & E_2 & E_1+E_4+E_7 & E_6 & & \\ & \ddots & \ddots & \ddots & & \\ & & & E_2 & E_1+E_4+E_7 & E_6 \\ & & & \mathbf{0} & E_2 & E_3+E_4+E_7 \end{bmatrix}.$$

Write $F_0 = E_1 + E_4 + E_5; F_1 = E_6; F_2 = E_2; F_3 = E_1 + E_4 + E_7; F_4 = E_3 + E_4 + E_7$.

$$\text{Then } \mathcal{A} = \begin{bmatrix} F_0 & F_1 & \mathbf{0} & & & \\ F_2 & F_3 & F_1 & & & \\ & F_2 & F_3 & F_1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & F_2 & F_3 & F_1 \\ & & & \mathbf{0} & F_2 & F_4 \end{bmatrix}.$$

Let $\boldsymbol{\pi} = [\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, ..., \boldsymbol{\pi}_{(N-1)}]$ denote the steady state probability vector of the generator matrix A
Here each $\boldsymbol{\pi}_i$'s are of dimension $1 \times m(k+n)$.

$$\boldsymbol{\pi}\mathcal{A} = 0, \boldsymbol{\pi}\boldsymbol{e} = 1. \tag{1}$$

From equation **(1)**, we get

$$\boldsymbol{\pi}_0 F_0 + \boldsymbol{\pi}_1 F_2 = \mathbf{0} \tag{2}$$

$$\boldsymbol{\pi}_0 F_1 + \boldsymbol{\pi}_1 F_3 + \boldsymbol{\pi}_2 F_2 = \mathbf{0} \tag{3}$$

$$\boldsymbol{\pi}_1 F_1 + \boldsymbol{\pi}_2 F_3 + \boldsymbol{\pi}_3 F_2 = \mathbf{0} \tag{4}$$

$$\boldsymbol{\pi}_2 F_1 + \boldsymbol{\pi}_3 F_3 + \boldsymbol{\pi}_4 F_2 = \mathbf{0} \tag{5}$$

$$\boldsymbol{\pi}_{(N-4)} F_1 + \boldsymbol{\pi}_{(N-3)} F_3 + \boldsymbol{\pi}_{(N-2)} F_2 = \mathbf{0} \tag{6}$$

$$\boldsymbol{\pi}_{(N-3)} F_1 + \boldsymbol{\pi}_{(N-2)} F_3 + \boldsymbol{\pi}_{(N-1)} F_2 = \mathbf{0} \tag{7}$$

$$\boldsymbol{\pi}_{(N-2)} F_1 + \boldsymbol{\pi}_{(N-1)} F_4 = \mathbf{0} \tag{8}$$

From equation **(8)**

$$\boldsymbol{\pi}_{(N-1)} = -\boldsymbol{\pi}_{(N-2)} F_1 F_4^{-1} \tag{9}$$

Take $U_{(N-2)} = -F_1 F_4^{-1}$

$$\boldsymbol{\pi}_{(N-1)} = \boldsymbol{\pi}_{(N-2)} U_{(N-2)} \tag{10}$$

From equation **(7)**

$$\boldsymbol{\pi}_{(N-2)} = -\boldsymbol{\pi}_{(N-3)} F_1 (F_3 + U_{(N-2)} F_2)^{-1} \tag{11}$$

Take $U_{(N-3)} = -F_1(F_3 + U_{(N-2)} F_2)^{-1}$

$$\boldsymbol{\pi}_{(N-2)} = \boldsymbol{\pi}_{(N-3)} U_{(N-3)} \tag{12}$$

Similarly from equation **(6)** we get,

$$\boldsymbol{\pi}_{(N-3)} = \boldsymbol{\pi}_{(N-4)} U_{(N-4)}, \tag{13}$$

where $U_{(N-4)} = -F_1(F_3 + U_{(N-3)} F_2)^{-1}$.

From equation **(3)** we get,

$$\boldsymbol{\pi}_2 = \boldsymbol{\pi}_1 U_1, \tag{14}$$

where $U_1 = -F_1(F_3 + U_2 F_2)^{-1}$ From equation **(2)** we get,

$$\boldsymbol{\pi}_1 = \boldsymbol{\pi}_0 U_0 \tag{15}$$

where $U_0 = -F_1(F_3 + U_1 F_2)^{-1}$.

$$\boldsymbol{\pi}_{(i+1)} = \boldsymbol{\pi}_i U_i, \text{where} \tag{16}$$

$$U_i = \begin{cases} -F_1(F_3 + U_{(i+1)} F_2)^{-1}, if\ 0 \le i \le (N-3) \\ -F_1 F_4^{-1}, if\ i = N-2 \end{cases}$$

$$\boldsymbol{\pi}_1 = \boldsymbol{\pi}_0 U_0 \tag{17}$$

$$\boldsymbol{\pi}_2 = \boldsymbol{\pi}_0 U_0 U_1 \tag{18}$$

$$\boldsymbol{\pi}_{N-2} = \boldsymbol{\pi}_0 U_0 U_1 U_2 .... U_{(N-3)} \tag{19}$$

$$\boldsymbol{\pi}_{N-1} = \boldsymbol{\pi}_0 U_0 U_1 U_2 .... U_{N-3} U_{N-2} = \boldsymbol{\pi}_0 \prod_{s=0}^{N-2} U_s. \tag{20}$$

Substituting the values of $\boldsymbol{\pi}_i$'s in the normalizing condition $\boldsymbol{\pi e} = 1$ we have,

$$\boldsymbol{\pi}_0 [I + \sum_{r=0}^{N-2} \prod_{s=0}^{r} U_s] \boldsymbol{e} = 1. \tag{21}$$

From equation **(21)**, we can find $\boldsymbol{\pi}_0$.

Hence we can find $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, ..., \boldsymbol{\pi}_{(N-1)}$.

3.1. Stability Condition

The LIQBD description of the model indicates that the queueing system is stable if and only if the left drift exceeds that of right drift. That is,

$$\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e}. \tag{22}$$

$$\boldsymbol{\pi} A_0 \mathbf{e} = \boldsymbol{\pi}_0 [I + \sum_{r=0}^{N-2} \prod_{s=0}^{r} U_s](I_{n+k} \otimes D_1)\boldsymbol{e} \tag{23}$$

$$\boldsymbol{\pi} A_2 \mathbf{e} = [\boldsymbol{\pi}_0 E_5 + \boldsymbol{\pi}_1 (E_7 + E_6) + \boldsymbol{\pi}_2 (E_7 + E_6) + \ldots\ldots + \boldsymbol{\pi}_{(N-2)} (E_7 + E_6) + \boldsymbol{\pi}_{(N-1)} E_7]\boldsymbol{e} \tag{24}$$

$$\boldsymbol{\pi} A_2 \mathbf{e} = \boldsymbol{\pi}_0 [E_5 + \sum_{r=0}^{N-2} \prod_{s=0}^{r} U_s E_7 + \sum_{r=0}^{N-3} \prod_{s=0}^{r} U_s E_6]\boldsymbol{e} \tag{25}$$

The stability condition is

$$\boldsymbol{\pi}_0 [I + \sum_{r=0}^{N-2} \prod_{s=0}^{r} U_s](I_{n+k} \otimes D_1)\boldsymbol{e} \leq \boldsymbol{\pi}_0 [E_5 + \sum_{r=0}^{N-2} \prod_{s=0}^{r} U_s E_7 + \sum_{r=0}^{N-3} \prod_{s=0}^{r} U_s E_6]\boldsymbol{e} \tag{26}$$

3.2. The Steady State Probability Vector of Q^*

Let $\boldsymbol{x}$ be the steady state probability vector of Q^*.

$\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2 \ldots)$, where $\boldsymbol{x}_0$ is of dimension $1 \times (Nm(k+n) + Nm)$ and $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ are each of dimension $1 \times Nm(k+n)$. Under the stability condition, we have $\boldsymbol{x}_i = \boldsymbol{x}_1 \mathcal{R}^{i-1}, i \geq 2$, where the matrix $\mathcal{R}$ is the minimal nonnegative solution to the matrix quadratic equation

$$\mathcal{R}^2 A_2 + \mathcal{R} A_1 + A_0 = 0$$

and the vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$are obtained by solving the equations

$$\boldsymbol{x}_0 B_0 + \boldsymbol{x}_1 B_1 = 0 \tag{27}$$

$$\boldsymbol{x}_0 B_0 + \boldsymbol{x}_1 (A_1 + \mathcal{R} A_2) = 0 \tag{28}$$

subject to the normalizing condition

$$\boldsymbol{x}_0 \boldsymbol{e} + \boldsymbol{x}_1 (I - \mathcal{R})^{-1} \boldsymbol{e} = 1 \tag{29}$$

4. Waiting Time Analysis

The server may be on vacation or in normal mode. So depending on the server's status, we obtain the expected waiting time of a particular customer by conditioning on the fact that at arrival epoch, the server is serving in vacation mode or normal mode.

4.1. Case 1-The server is in vacation mode

To find the expected waiting time of a tagged customer who joins as the r^{th} customer in the queue, we consider the Markov Processes $W_v(t) = \{(\mathcal{N}(t), M(t), J(t), S(t)) : t \geq 0\}$ where

$\mathcal{N}(t)$: Rank of the customer in the queue at time t.
$M(t)$: Number of customers in the buffer at time t.

$$J(t) = \begin{cases} 0, & \text{if the server is in vacation mode at time } t. \\ 1, & \text{if the server is in normal mode at time } t. \end{cases}$$

$S(t)$: Phase of the service at time t.
The rank of the customer decrease by one when a customer ahead of him completes the service. State space of $W_v(t)$ is
$\Omega_1 = \{\{r, r-1, r-2, \cdots, 2, 1\} \times \{0, 1, 2, 3, \cdots, N-1\} \times \{0\} \times \{1, 2, , \cdots, k\}\} \cup \{\{r, r-1, r-2, \cdots, 2, 1\} \times \{0, 1, 2, 3, \cdots, N-1\} \times \{1\} \times \{1, 2, 3 \cdots, n\}\} \cup \{\Delta\}$, where Δ denotes the absorbing state - beginning of the preliminary service of the tagged customer.

The infinitesimal generator is

$\mathcal{Q}_1 = \begin{bmatrix} W & W^0 \\ \mathbf{0} & 0 \end{bmatrix}$, where, $W = \begin{bmatrix} G & H & & & \\ & G & H & & \\ & & \ddots & \ddots & \\ & & & G & H \\ & & & & G \end{bmatrix}$ is a square matrix of order $N(n+k)r$ and $W^0 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \\ \mathbf{0} \\ H \end{bmatrix}$ is a matrix of order $N(n+k)r \times 1$.

$$G = \begin{bmatrix} G_1 & \mathbf{0} & & & & \\ G_2 & G_1 & & & & \\ & G_2 & G_1 & & & \\ & & \ddots & \ddots & & \\ & & & G_2 & G_1 & \\ & & & & G_2 & G_3 \end{bmatrix}$$ is a square matrix of order $N(n+k)$.

$$G_1 = \begin{bmatrix} G_{11} & G_{12} \\ \mathbf{0} & G_{13} \end{bmatrix}; G_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & G_{21} \end{bmatrix}; G_3 = \begin{bmatrix} G_{11} & G_{14} \\ \mathbf{0} & G_{13} \end{bmatrix}.$$

$$G_{11} = \begin{bmatrix} -(\theta+\eta) & \theta & & \\ & -(\theta+\eta) & \theta & \\ & & \ddots & \ddots \\ & & & -(\theta+\eta) \end{bmatrix}$$ is a square matrix of order k.

$G_{12} = \begin{bmatrix} \eta I_k & \mathbf{0} \end{bmatrix}$ is a $k \times n$ matrix.

$$G_{13} = \begin{bmatrix} -\theta & \theta & & & & & & & \\ & -\theta & \theta & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & -\theta & \theta & & & & \\ & & & & -\phi & \phi & & & \\ & & & & & -\phi & \phi & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & & -\phi \end{bmatrix} \text{ is a square matrix of order } n.$$

$G_{14} = \begin{bmatrix} \eta I_k & \boldsymbol{e}_k(k)\theta(1-p) & \mathbf{0} \end{bmatrix}$ is a matrix of order $k \times n$.

$G_{21} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \overline{\boldsymbol{e}}_{k+1}(k+1)\phi & \mathbf{0} \end{bmatrix}$ is a matrix of order $k \times n$.

$$H = \begin{bmatrix} H_1 & H_2 & & & \\ & H_3 & H_2 & & \\ & & \ddots & \ddots & \\ & & & H_3 & H_2 \\ & & & & H_3 \end{bmatrix} \text{ is a square matrix of order } N(n+k).$$

$$H_1 = \begin{bmatrix} H_{11} & \mathbf{0} \\ \mathbf{0} & H_{12} \end{bmatrix}; H_2 = \begin{bmatrix} H_{21} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; H_3 = \begin{bmatrix} H_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

$H_{11} = \begin{bmatrix} \boldsymbol{e}_k(k)\theta p & \mathbf{0} \end{bmatrix}$ is a square matrix of order k.

$H_{12} = \begin{bmatrix} \boldsymbol{e}_n(n)\phi & \mathbf{0} \end{bmatrix}$ is a square matrix of order n.

$H_{21} = \begin{bmatrix} \boldsymbol{e}_k(k)\theta(1-p) & \mathbf{0} \end{bmatrix}$ is a square matrix of order k.

The initial probability vector is $\boldsymbol{\beta} = \overline{\boldsymbol{e}}_1(N(n+k)r)$.
Waiting time W_v of a customer, who joins the queue as the rth customer is the time until absorption. Therefore expected waiting time of the particular customer is

$E^r_{W_v} = \boldsymbol{\beta}(-W)^{-1}\mathbf{e}$.
Expected waiting time of a general customer = $\Sigma_{r=1}^{\infty}\boldsymbol{x}_r E^r_{W_v}$.

4.2. Case 2-The server is in normal mode

To find the expected waiting time of a particular customer who joins as the rth customer in the queue, we consider the Markov Processes $W_n(t) = \{(N(t), M(t), S(t)) : t \geq 0\}$ where
$N(t)$- Rank of the customer in the queue at time t.
$M(t)$- Number of customers in the buffer at time t.
$S(t)$- Phase of the service at time t.
The state space of the Markov Process is
$\{r, r-1, r-2, \cdots, 2, 1\} \times \{0, 1, 2, 3, \cdots, (N-1)\} \times \{1, 2, 3 \cdots, n\} \cup \{\Delta_1\}$
where Δ_1 denote the absorbing state - begining of the preliminary service of the tagged customer.
The infinitesimal generator is

$$\mathcal{Q}_2 = \begin{bmatrix} W_1 & W_1^0 \\ \mathbf{0} & 0 \end{bmatrix}, \text{where, } W_1 = \begin{bmatrix} W_{11} & \mathbf{0} & \mathbf{0} & & \\ \mathbf{0} & G_{13} & H_{12} & & \\ & & \ddots & \ddots & \\ & & & G_{13} & H_{12} \\ & & & & G_{13} \end{bmatrix}, \text{is a square matrix}$$

of order $N(n-k) + rn$.

$$W_1^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \phi \end{bmatrix} \text{ is a } [N(n-k) + rn] \times 1 \text{ matrix.}$$

$$T_{11} = \begin{bmatrix} -\phi & \phi & & & & \\ & -\phi & \phi & & & \\ & & -\phi & \phi & & \\ & & & -\phi & \phi & \\ & & & & \ddots & \ddots \\ & & & & & -\phi \end{bmatrix} \text{ is a square matrix of order } N(n-k).$$

$$T_{12} = \begin{bmatrix} e_{N(n-k)}(N(n-k))\phi & \mathbf{0} \end{bmatrix} \text{ is a matrix of order } N(n-k) \times n.$$

$$W_{11} = \begin{bmatrix} T_{11} & T_{12} \\ \mathbf{0} & G_{13} \end{bmatrix} \text{ is asquare matrix of order } N(n-k) + n.$$

The initial probability vector is $\boldsymbol{\gamma} = \bar{\boldsymbol{e}}_1(N(n-k) + nr)$.
Expected waiting time of the tagged customer $E_{W_n}^r = \boldsymbol{\gamma}(-W_1)^{-1}\mathbf{e}$.

Expected waiting time of a general customer$=\Sigma_{r=1}^{\infty} \boldsymbol{x}_r E_{W_n}^r$.

5. Additional Performance Measures

- Probability that the server is idle:

$$P_{idle} = \sum_{j=1}^{m} \boldsymbol{x}_{000*j} + \sum_{h=0}^{N-1} \sum_{j=1}^{m} \boldsymbol{x}_{0h0*j}.$$

- Probability that the system is in vacation mode:

$$P_{vacation} = \sum_{q=1}^{\infty} \sum_{h=0}^{N-1} \sum_{i=1}^{k} \sum_{j=1}^{m} \boldsymbol{x}_{qh0ij} + \sum_{h=0}^{N-1} \sum_{j=1}^{m} \boldsymbol{x}_{0h0*j} + \sum_{j=1}^{m} \boldsymbol{x}_{000*j} + \sum_{h=0}^{N-1} \sum_{i=1}^{k} \sum_{j=1}^{m} \boldsymbol{x}_{0h0ij}$$

- Probability that system is in normal mode:

$$P_{normal} = \sum_{q=1}^{\infty} \sum_{h=0}^{N-1} \sum_{i=1}^{n} \sum_{j=1}^{m} \boldsymbol{x}_{qh1ij}$$

- Probability that no customers in the queue

$$P_0 = \boldsymbol{x}_0 \boldsymbol{e}.$$

- Probability that there are q customers in the queue:

$$P_q = \boldsymbol{x}_q \boldsymbol{e}.$$

- Expected number of customers in the queue:

$$ECQ = \sum_{q=1}^{\infty} q \boldsymbol{x}_q \boldsymbol{e}$$

- Expected number of customers in the system:

$$ECS = \sum_{q=1}^{\infty} (q+1) \boldsymbol{x}_q \boldsymbol{e}$$

- Expected number of customers in the buffer:

$$ECB = \sum_{q=0}^{\infty} \sum_{h=0}^{N-1} \sum_{i=1}^{n} \sum_{j=1}^{m} h \boldsymbol{x}_{qh1ij} + \sum_{q=0}^{\infty} \sum_{h=0}^{N-1} \sum_{i=1}^{k} \sum_{j=1}^{m} h \boldsymbol{x}_{qh0ij}$$

- Rate of switching to normal mode

$$RN = \sum_{q=0}^{\infty} \sum_{h=0}^{N-1} \sum_{i=1}^{k} \sum_{j=1}^{m} \boldsymbol{x}_{qh0ij} \eta + \sum_{q=0}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \boldsymbol{x}_{q(N-1)0ij} k\theta(1-p)$$

6. Cost Function

To find optimal N, we construct a cost function as follows.
Let
CV – Cost per unit time when service is in vacation mode.
CN – Cost per unit time when service is in normal mode.
HCQ – Holding cost per customer in the queue.
CSN – Cost per unit time for switching to normal mode.
HCB – Holding cost per cusomer in buffer.
Then expected cost is,

$$\begin{aligned} EC &= k\theta \times CV \times P_{vac} + [k\theta + (n-k)\phi] \times CN \times P_{normal} + ECQ \times HCQ \\ &\quad + CSN \times RN + HCB \times ECB. \end{aligned}$$

We take $HCQ = 4$, $CSN = 200$, $HCB = 5$, $CV = 8$, $CN = 10$.

7. Numerical Results

For the arrival process of customers, we consider the following three sets of matrices for D_0 and D_1

1. MAP with positive correlation (MPC):
$$D_0 = \begin{bmatrix} -2.0151 & 2.0151 & 0 \\ 0 & -2.2787 & 0 \\ 0 & 0 & -59.8481 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2.1996 & 0 & 0.0791 \\ 2.0773 & 0 & 57.7708 \end{bmatrix}.$$
2. MAP with negative correlation (MNC):
$$D_0 = \begin{bmatrix} -2.0151 & 2.0151 & 0 \\ 0 & -2.2787 & 0 \\ 0 & 0 & -59.8481 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.0791 & 0 & 2.1996 \\ 57.7708 & 0 & 2.0773 \end{bmatrix}.$$
3. MAP with zero correlation (MZC):
$$D_0 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2.48 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.95 & 0 & 0.05 \\ 0.18 & 0 & 2.3 \end{bmatrix}.$$

The arrival process labelled MNC has correlated arrivals with the correlation between two successive interarrival times given by -0.4559, the arrival process corresponding to the one labelled MPA has correlated arrivals with the correlation between two successive interarrival times given by 0.4559 and the arrival process labeled MZC has zero correlation between two successive interarrival times. The arrival rate in all the above three cases is $\lambda = 2.011$.

7.1. MAP with positive correlation (MPC)

Table 1. Effect of θ: Fix $n = 5, N = 4, k = 2, m = 3$, $\phi = 12, \eta = 5, p = 0.1$.

θ	ECQ	ECS	ECB	P_{idle}	P_{normal}	P_{vac}	RN	EC
10	208.4606	210.2758	0.9200	0.0611	0.9292	0.0708	0.0491	1379.90
11	128.5954	130’3095	0.8747	0.0960	0.8896	0.1104	0.0729	1068.70
12	93.7787	95.4046	0.8343	0.1273	0.8547	0.1453	0.0917	938.3411
13	74.9243	76.4755	0.7996	0.1544	0.8248	0.1752	0.1059	872.7115
14	63.2334	64.7213	0.7696	0.1779	0.7993	0.2007	0.1166	836.6233
15	55.3068	56.7406	0.7437	0.1984	0.7775	0.2225	0.1245	816.3720
16	49.5875	50.9746	0.7211	0.2164	0.7585	0.2415	0.1303	805.6202
17	45.2690	46.6155	0.7012	0.2322	0.7419	0.2581	0.1346	801.0482
18	41.8938	43.2047	0.6836	0.2464	0.7274	0.2726	0.1377	**800.7470**
19	39.1837	40.4630	0.6679	0.2590	0.7144	0.2856	0.1398	803.5381
20	36.9599	38.2111	0.6538	0.2704	0.7029	0.2971	0.1413	808.6515
21	35.1023	36.3283	0.6411	0.2808	0.6926	0.3074	0.1422	815.5612
22	33.5274	34.7306	0.6296	0.2902	0.6833	0.3167	0.1427	823.8940
23	32.1752	33.3578	0.6191	0.2988	0.6748	0.3252	0.1428	833.3772
24	31.0015	32.1654	0.6095	0.3067	0.6671	0.3329	0.1427	843.8059

Table 2. Effect of ϕ: Fix $n = 5, N = 4, k = 2, m = 3$, $\theta = 10, \eta = 5, p = 0.1$.

ϕ	ECQ	ECS	ECB	P_{idle}	P_{normal}	P_{vac}	RN	EC
12	208.4606	210.2758	0.9200	0.0611	0.9292	0.0708	0.0491	1379.90
13	125.5585	127.2693	0.8747	0.0979	0.8866	0.1134	0.0782	1063.5
14	89.7402	91.3587	0.8347	0.1317	0.8475	0.1525	0.1047	933.9435
15	70.4792	72.0192	0.8006	0.1617	0.8130	0.1870	0.1279	869.8836
16	58.5898	60.0631	0.7715	0.1880	0.7827	0.2173	0.1481	834.8472
17	50.5517	51.9679	0.7466	0.2112	0.7559	0.2441	0.1659	814.8786
18	44.7629	46.1296	0.7251	0.2319	0.7322	0.2678	0.1815	803.6424
19	40.3978	41.7214	0.7063	0.2504	0.7110	0.2890	0.1954	797.8854
20	36.9896	38.2752	0.6897	0.2670	0.6919	0.3081	0.2078	**795.7771**
21	34.2551	35.5071	0.6750	0.2820	0.6747	0.3253	0.2189	796.2129
22	32.0127	33.2348	0.6619	0.2957	0.6590	0.3410	0.2290	798.4886
23	30.1406	31.3358	0.6502	0.3081	0.6448	0.3552	0.2381	802.1354
24	28.5542	29.7251	0.6396	0.3195	0.6317	0.3683	0.2464	806.8294
25	27.1926	28.3416	0.6300	0.3300	0.6198	0.3802	0.2540	812.3401
26	26.0114	27.1404	0.6212	0.3396	0.6087	0.3913	0.2610	818.4992

Tables 1 to 4 contain the effect of different parameters on various performance measures and the cost function when the arrival process of the customers is MPC. Table 1 indicates the effect of θ on various performance measures and the cost function. When the values of θ(service rate of the first part of the service) increase, the values of ECS, ECQ, and ECB decrease. It is because the expected service time in the first stage of the service decreases.

Table 3. Effect of η: Fix $n = 5, N = 4, k = 2, m = 3$, $\theta = 14, \phi = 15, p = 0.1$.

η	ECQ	ECS	ECB	P_{idle}	P_{normal}	P_{vac}	RN	EC
5	34.2249	35.4547	0.6529	0.2838	0.6806	0.3194	0.1831	745.1509
6	34.2239	35.4505	0.6500	0.2833	0.6836	0.3164	0.2012	750.3051
7	34.2239	35.4483	0.6480	0.2830	0.6863	0.3137	0.2171	754.8219
8	34.2243	35.4472	0.6466	0.2827	0.6886	0.3114	0.2310	758.7844
9	34.2248	35.4466	0.6455	0.2824	0.6907	0.3093	0.2433	762.2725
10	34.2255	35.4464	0.6447	0.2822	0.6925	0.3075	0.2541	765.3560
11	34.2261	35.4464	0.6440	0.2819	0.6941	0.3059	0.2637	768.0940
12	34.2267	35.4465	0.6435	0.2818	0.6956	0.3044	0.2722	770.5358
13	34.2273	35.4466	0.6431	0.2816	0.6969	0.3031	0.2798	772.7227
14	34.2278	35.4468	0.6428	0.2814	0.6981	0.3019	0.2866	774.6892
15	34.2283	35.4471	0.6425	0.2813	0.6992	0.3008	0.2928	776.4643
16	34.2288	35.4473	0.6423	0.2812	0.7002	0.2998	0.2983	778.0722
17	34.2292	35.4475	0.6421	0.2811	0.7011	0.2989	0.3033	779.5336
18	34.2296	35.4477	0.6420	0.2810	0.7019	0.2981	0.3079	780.8662
19	34.2299	35.4479	0.6418	0.2809	0.7027	0.2973	0.3120	782.0849

Table 4. Effect of p: Fix $n = 5, N = 4, k = 2, m = 3$, $\theta = 14, \eta = 5, \phi = 15$.

p	ECQ	ECS	ECB	P_{idle}	P_{normal}	P_{vac}	RN	EC
0.1	34.2249	35.4547	0.6529	0.2838	0.6806	0.3194	0.1831	745.1509
0.15	34.2059	35.4341	0.6518	0.2859	0.6782	0.3218	0.1837	743.9812
0.2	34.1866	35.4130	0.6507	0.2881	0.6758	0.3242	0.1843	742.8013
0.25	34.1669	35.3916	0.6496	0.2903	0.6733	0.3267	0.1849	741.6114
0.3	34.1468	35.3697	0.6485	0.2926	0.6709	0.3291	0.1856	740.4118
0.35	34.1263	35.3475	0.6474	0.2948	0.6684	0.3316	0.1863	739.2029
0.4	34.1056	35.3249	0.6462	0.2971	0.6659	0.3341	0.1870	737.9852
0.45	34.0845	35.3020	0.6450	0.2994	0.6633	0.3367	0.1878	736.7592
0.5	34.0631	35.2787	0.6438	0.3018	0.6607	0.3393	0.1886	735.5254
0.55	34.0415	35.2552	0.6426	0.3042	0.6581	0.3419	0.1895	734.2845
0.6	34.0197	35.2315	0.6413	0.3066	0.6555	0.3445	0.1904	733.0370
0.65	33.9977	35.2075	0.6400	0.3090	0.6528	0.3472	0.1913	731.7833
0.7	33.9755	35.1834	0.6387	0.3115	0.6501	0.3499	0.1923	730.5237

The value of the EC decreases to reach the minimum value at $\theta = 18$, and after that, the values increase. The minimum value of the cost function, in this case, is 800.7470. P_{vac}, P_{idle}, and RN increase when the θ values increase. But P_{nor} decreases when θ increases.

Table 2 indicates the effect of ϕ on various performance measures and the cost function. When the values of ϕ(service rate of the second part of the service) increase, the values of ECS, ECQ, and ECB decrease. It is because the expected service time in the main service decreases. The value of the EC decreases to reach the minimum value at $\phi = 20$, and then the value increases. The minimum cost, in this case, is 795.7771. P_{vac}, P_{idle}, and RN increase when the ϕ value increases since the expected service rate in main services increases. But P_{nor} decreases when ϕ increases.

Table 3 indicates the effect of η on various performance measures and the cost function. As η increases, the server turns to normal mode quickly. So the values of P_{vac} decrease. When the values of η increase, there are only very small changes in the values of ECS, ECQ, ECB, and P_{idle}. The value of the EC increases when the value η increases. P_{nor} and RN also increase, since the vacation realizes speedily when η increases.

Table 4 indicates the effect of p on various performance measures and the cost function. When the p value increases, the number of customers who leave the system after availing of the first part of the service increases. So the values of ECB, EC and, P_{normal} decrease. But the values of P_{idle}, P_{vac}, and RN decrease when p increases.

7.2. MAP with negative correlation (MNC)

Table 5. Effect of θ: Fix $n = 5, N = 4, k = 2, m = 3, \phi = 12, \eta = 5, p = 0.1$.

θ	ECQ	ECS	ECB	P_{idle}	P_{normal}	P_{vac}	RN	EC
10	10.1840	11.9983	0.9283	0.0559	0.9299	0.0701	0.0742	592.1588
11	5.6622	7.3499	0.8772	0.0953	0.8814	0.1186	0.1222	**583.5764**
12	4.0358	5.6198	0.8338	0.1282	0.8417	0.1583	0.1592	587.5558
13	3.2048	4.7022	0.7964	0.1562	0.8086	0.1914	0.1879	595.5141
14	2.7029	4.1268	0.7637	0.1803	0.7807	0.2193	0.2103	605.4496
15	2.3682	3.7286	0.7349	0.2013	0.7569	0.2431	0.2279	616.6098
16	2.1295	3.4347	0.7092	0.2196	0.7363	0.2637	0.2418	628.6294
17	1.9511	3.2076	0.6862	0.2359	0.7185	0.2815	0.2527	641.2997
18	1.8128	3.0260	0.6654	0.2504	0.7029	0.2971	0.2613	654.4880
19	1.7025	2.8771	0.6465	0.2635	0.6891	0.3109	0.2680	668.1027
20	1.6126	2.7524	0.6293	0.2752	0.6769	0.3231	0.2732	682.0770
21	1.5380	2.6462	0.6135	0.2859	0.6660	0.3340	0.2771	696.3600
22	1.4750	2.5546	0.5989	0.2902	0.6562	0.3438	0.2801	710.9118
23	1.4212	2.4746	0.5855	0.3044	0.6474	0.3526	0.2822	725.7001
24	1.3747	2.4040	0.5731	0.3126	0.6395	0.3605	0.2837	740.6984

Tables 5 to 8 contain the effect of different parameters on various performance measures and the cost function when the arrival process of the customers is MNC. Table 5 indicates the effect of θ on various performance measures and the cost function. When the values of θ(service rate of the first part of the service) increase, the values of ECS, ECQ, and ECB decrease. It is because the expected service time in the first stage of the service decreases. The value of the EC decreases to reach the minimum value at $\theta = 11$, and after that, the values increase. In this case, the cost function's minimum value is 583.5764. P_{vac}, P_{idle}, and RN increase when the θ values increase. But P_{nor} decreases when θ increases.

Table 6 indicates the effect of ϕ on various performance measures and the cost function. When the values of ϕ(service rate of the second part of the service) increase, the values of ECS, ECQ, and ECB decrease. It is because the expected service time in the second part of the service decreases. The value of the EC decreases to reach the minimum value at $\phi = 13$,

Table 6. Effect of ϕ: Fix $n = 5, N = 4, k = 2, m = 3, \theta = 10, \eta = 5, p = 0.1$.

ϕ	ECQ	ECS	ECB	P_{idle}	P_{normal}	P_{vac}	RN	EC
12	10.1840	11.9983	0.9283	0.0559	0.9299	0.0701	0.0742	592.1588
13	5.5247	7.2121	0.8790	0.0971	0.8783	0.1217	0.1280	**589.7377**
14	3.8671	5.4526	0.8393	0.1323	0.8343	0.1657	0.1732	598.0930
15	3.0242	4.5264	0.8068	0.1628	0.7965	0.2035	0.2116	608.7482
16	2.5167	3.9497	0.7796	0.1893	0.7636	0.2364	0.2447	619.9843
17	2.1788	3.5537	0.7567	0.2127	0.7347	0.2653	0.2735	631.2870
18	1.9384	3.2636	0.7371	0.2334	0.7091	0.2909	0.2987	642.4799
19	1.7588	3.0413	0.7202	0.2519	0.6864	0.3136	0.3210	653.5043
20	1.6198	2.8652	0.7055	0.2685	0.6659	0.3341	0.3408	664.3472
21	1.5090	2.7219	0.6926	0.2835	0.6475	0.3525	0.3585	675.0140
22	1.4187	2.6029	0.6812	0.2971	0.6308	0.3692	0.3744	685.5173
23	1.3437	2.5024	0.6710	0.3096	0.6156	0.3844	0.3888	695.8718
24	1.2805	2.4164	0.6619	0.3210	0.6017	0.3983	0.4019	706.0924
25	1.2264	2.3419	0.6537	0.3314	0.5889	0.4111	0.4138	716.1929
26	1.1797	2.2766	0.6463	0.3411	0.5772	0.4228	0.4247	726.1859

Table 7. Effect of η: Fix $n = 5, N = 4, k = 2, m = 3, \theta = 14, \phi = 15, p = 0.1$.

η	ECQ	ECS	ECB	P_{idle}	P_{normal}	P_{vac}	RN	EC
5	1.4859	2.6407	0.6513	0.2868	0.6525	0.3475	0.3270	628.7896
6	1.4838	2.6343	0.6441	0.2858	0.6599	0.3401	0.3396	634.9674
7	1.4830	2.6315	0.6393	0.2851	0.6659	0.3341	0.3514	640.3678
8	1.4829	2.6308	0.6361	0.2844	0.6710	0.3290	0.3617	644.9881
9	1.4831	2.6311	0.0.6339	0.2839	0.6753	0.3247	0.3705	648.9134
10	1.4835	2.6320	0.6323	0.2834	0.6790	0.3210	0.3779	652.2472
11	1.4840	2.6333	0.6312	0.2831	0.6821	0.3179	0.3842	655.0871
12	1.4845	2.6347	0.6304	0.2827	0.6849	0.3151	0.3894	657.5175
13	1.4850	2.6361	0.6297	0.2825	0.6873	0.3127	0.3937	659.6086
14	1.4855	2.6376	0.6293	0.2822	0.6894	0.3106	0.3974	661.4182
15	1.4859	2.6389	0.6289	0.2820	0.6913	0.3087	0.4005	662.9931
16	1.4864	2.6403	0.6286	0.2818	0.6930	0.3070	0.4031	664.3715
17	1.4868	2.6415	0.6284	0.2816	0.6945	0.3055	0.4053	665.5846
18	1.4871	2.6427	0.6283	0.2815	0.6959	0.3041	0.4072	666.6576
19	1.4875	2.6438	0.6282	0.2814	0.6971	0.3029	0.4089	667.6117

and then the value increases. The minimum cost, in this case, is 589.7377. P_{vac}, P_{idle}, and RN increase when the ϕ value increases since the expected service rate in main services increases. But P_{nor} decreases when ϕ increases.

Table 7 indicates the effect of η on various performance measures and the cost function. As η increases, the server turns to normal mode quickly. So the values of P_{vac} decrease. When the values of the η increase, there are only very small changes in the values of ECS, ECQ, ECB, and P_{idle}. The value of the EC increases when the value η increases. P_{nor} and

Table 8. Effect of p: Fix $n = 5, N = 4, k = 2, m = 3, \theta = 14, \eta = 5, \phi = 15$.

p	ECQ	ECS	ECB	P_{idle}	P_{normal}	P_{vac}	RN	EC
0.1	1.4859	2.6407	0.6513	0.2868	0.6525	0.3475	0.3270	628.7896
0.15	1.4747	2.6218	0.6463	0.2905	0.6481	0.3519	0.3265	626.3916
0.2	1.4633	2.6026	0.6413	0.2943	0.6436	0.3564	0.3265	624.0468
0.25	1.4518	2.5832	0.6361	0.2982	0.6391	0.3609	0.3270	621.7476
0.3	1.4402	2.5636	0.6309	0.3022	0.6344	0.3656	0.3279	619.4874
0.35	1.4285	2.5437	0.6255	0.3062	0.6296	0.3704	0.3293	617.2598
0.4	1.4167	2.5236	0.6201	0.3104	0.6247	0.3753	0.3310	615.0586
0.45	1.4048	2.5034	0.6146	0.3147	0.6197	0.3803	0.3331	612.8778
0.5	1.3927	2.4829	0.6091	0.3190	0.6146	0.3854	0.3356	610.7114
0.55	1.3806	2.4623	0.6035	0.3235	0.6094	0.3906	0.3384	608.532
0.6	1.3684	2.4415	0.5978	0.3280	0.6040	0.3960	0.3415	606.3968
0.65	1.3562	2.4205	0.5921	0.3327	0.5986	0.4014	0.3449	604.2355
0.7	1.3438	2.3995	0.5863	0.3374	0.5930	0.4070	0.3485	602.0623

RN also increase, since the vacation realizes speedily when η increases.

Table 8 indicates the effect of p on various performance measures and the cost function. When the p value increases, the number of customers who leave the system after availing of the first part of the service increases. So the values of ECB, EC, and P_{normal} decrease. But the values of P_{idle}, P_{vac}, and RN decrease.

7.3. MAP with zero correlation (MZC)

Table 9. Effect of θ: Fix $n = 5, N = 4, k = 2, m = 3, \phi = 12, \eta = 5, p = 0.1$.

θ	ECQ	ECS	ECB	P_{idle}	P_{normal}	P_{vac}	RN	EC
10	12.6023	14.4141	0.9271	0.0554	0.9325	0.0675	0.0620	600.4481
11	6.8200	8.5020	0.8750	0.0947	0.8854	0.1146	0.1029	**585.9489**
12	4.7560	6.3319	0.8309	0.1275	0.8467	0.1533	0.1344	587.5102
13	3.7096	5.1971	0.7931	0.1554	0.8143	0.1857	0.1588	594.0701
14	3.0826	4.4955	0.7603	0.1793	0.7870	0.2130	0.1777	603.0575
15	2.6676	4.0166	0.7316	0.2002	0.7636	0.2364	0.1923	613.5131
16	2.3740	3.6679	0.7062	0.2185	0.7435	0.2565	0.2036	624.9775
17	2.1561	3.4018	0.6835	0.2346	0.7259	0.2741	0.2122	637.1930
18	1.9884	3.1917	0.6632	0.2490	0.7106	0.2894	0.2188	649.9982
19	1.8556	3.0213	0.6449	0.2620	0.6970	0.3030	0.2236	663.2840
20	1.7480	2.8801	0.6284	0.2736	0.6850	0.3150	0.2272	676.9720
21	1.6591	2.7611	0.6132	0.2842	0.6743	0.3257	0.2296	691.0033
22	1.5846	2.6594	0.5994	0.2938	0.6646	0.3354	0.2312	705.3321
23	1.5212	2.5714	0.5868	0.3026	0.6560	0.3440	0.2321	719.9220
24	1.4667	2.4945	0.5751	0.3106	0.6481	0.3519	0.2324	734.7429

Table 10. Effect of ϕ: Fix $n = 5, N = 4, k = 2, m = 3, \theta = 10, \eta = 5, p = 0.1$.

ϕ	ECQ	ECS	ECB	P_{idle}	P_{normal}	P_{vac}	RN	EC
12	12.6023	14.4141	0.9271	0.0554	0.9325	0.0675	0.0620	600.4481
13	6.6390	8.3177	0.8759	0.0966	0.8820	0.1180	0.1097	**592.1543**
14	4.5339	6.1039	0.8341	0.1318	0.8387	0.1613	0.1515	598.4154
15	3.4721	4.9518	0.7995	0.1623	0.8011	0.1989	0.1884	608.1076
16	2.8382	4.2422	0.7705	0.1888	0.7682	0.2318	0.2212	618.8830
17	2.4200	3.7596	0.7459	0.2122	0.7391	0.2609	0.2504	629.9858
18	2.1250	3.4094	0.7249	0.2330	0.7132	0.2868	0.2767	641.1229
19	1.9066	3.1432	0.7067	0.2515	0.6901	0.3099	0.3003	652.1720
20	1.7389	2.9338	0.6908	0.2682	0.6692	0.3308	0.3218	663.0823
21	1.6064	2.7647	0.6769	0.2832	0.6504	0.3496	0.3413	673.8361
22	1.4993	2.6252	0.6646	0.2969	0.6333	0.3667	0.3591	684.4311
23	1.4110	2.5081	0.6537	0.3093	0.6176	0.3824	0.3755	694.8728
24	1.3370	2.4083	0.6439	0.3207	0.6033	0.3967	0.3905	705.1699
25	1.2742	2.3223	0.6351	0.3312	0.5901	0.4099	0.4044	715.3324
26	1.2202	2.2474	0.6272	0.3409	0.5779	0.4221	0.4173	725.3706

Table 11. Effect of η: Fix $n = 5, N = 4, k = 2, m = 3, \theta = 14, \phi = 15, p = 0.1$.

η	ECQ	ECS	ECB	P_{idle}	P_{normal}	P_{vac}	RN	EC
5	1.5893	2.7084	0.6406	0.2860	0.6586	0.3414	0.2922	625.2544
6	1.5841	2.6937	0.6313	0.2852	0.6644	0.3356	0.3105	631.8012
7	1.5808	2.6840	0.6248	0.2846	0.6693	0.3307	0.3279	637.6762
8	1.5786	2.6674	0.6202	0.2841	0.6743	0.3266	0.3437	642.8655
9	1.5772	2.6727	0.6167	0.2836	0.6768	0.3232	0.3578	647.4312
10	1.5761	2.6693	0.6141	0.2832	0.6798	0.3202	0.3704	651.4529
11	1.5754	2.6667	0.6121	0.2829	0.6825	0.3175	0.3815	655.0075
12	1.5749	2.6648	0.6105	0.2826	0.6848	0.3152	0.3915	658.1629
13	1.5746	2.6634	0.6092	0.2824	0.6869	0.3131	0.4004	660.9767
14	1.5743	2.6622	0.6082	0.2821	0.6887	0.3113	0.4084	663.4974
15	1.5741	2.6614	0.6073	0.2819	0.6904	0.3096	0.4155	665.7654
16	1.5740	2.6607	0.6066	0.2818	0.6919	0.3081	0.4220	667.8145
17	1.5739	2.6601	0.6060	0.2816	0.6932	0.3068	0.4279	669.6731
18	1.5738	2.6597	0.6055	0.2815	0.6945	0.3055	0.4332	671.3649
19	1.5738	2.6593	0.6050	0.2813	0.6956	0.3044	0.4381	672.9103

Tables 9 to 12 contain the effect of different parameters on various performance measures and the cost function when the arrival process of the customers is MZC. Table 9 indicates the effect of θ on various performance measures and the cost function. When the values of θ(service rate of the first part of the service) increase, the values of ECS, ECQ, and ECB decrease. It is because the expected service time in the first stage of the service decreases. The value of the EC decreases to reach the minimum value at $\theta = 11$, and after that, the values increase. The minimum value of the cost function, in this case, is 585.9489.

Table 12. Effect of p: Fix $n = 5, N = 4, k = 2, m = 3, \theta = 14, \eta = 5, \phi = 15$.

p	ECQ	ECS	ECB	P_{idle}	P_{normal}	P_{vac}	RN	EC
0.1	1.5893	2.7084	0.6406	0.2860	0.6586	0.3414	0.2922	625.2544
0.15	1.5785	2.6901	0.6363	0.2894	0.6544	0.3456	0.2938	623.3943
0.2	1.5675	2.6715	0.6319	0.2929	0.6501	0.3499	0.2958	621.5304
0.25	1.5563	2.6526	0.6274	0.2966	0.6457	0.3543	0.2980	619.6617
0.3	1.5449	2.6333	0.6228	0.3003	0.6411	0.3589	0.3005	617.7870
0.35	1.5333	2.6137	0.6181	0.3041	0.6364	0.3636	0.3033	615.9045
0.4	1.5215	2.5937	0.6133	0.3081	0.6316	0.3684	0.3064	614.0122
0.45	1.5095	2.5734	0.6085	0.3122	0.6266	0.3734	0.3099	612.1075
0.5	1.4973	2.5527	0.6035	0.3164	0.6215	0.3785	0.3136	610.1871
0.55	1.4849	2.5316	0.5985	0.3207	0.6162	0.3838	0.3177	608.2472
0.6	1.4723	2.5102	0.5934	0.3251	0.6107	0.3893	0.3220	606.2834
0.65	1.4594	2.4884	0.5882	0.3297	0.6051	0.3949	0.3267	604.2906
0.7	1.4463	2.4661	0.5829	0.3345	0.5993	0.4007	0.3317	602.2627

P_{vac}, P_{idle}, and RN increase when the θ values increase. But P_{nor} decreases when the θ increases.

Table 10 indicates the effect of ϕ on various performance measures and the cost function. When the values of ϕ(service rate of the second part of the service) increase, the values of ECS, ECQ, and ECB decrease. It is because the expected service time in the main service decreases. The value of the EC decreases to reach the minimum value at $\phi = 13$ and then the value increases. The minimum cost, in this case, is 592.1543. P_{vac}, P_{idle}, and RN increase when the ϕ value increases since the expected service rate in main services increases. But P_{nor} decreases when ϕ increases.

Table 11 indicates the effect of η on various performance measures and the cost function. As η increases, the server turns to normal mode quickly. So the values of P_{vac} decrease. When the values of the η increase, there are only very small changes in the values of ECS, ECQ, ECB, and P_{idle}. The value of the EC increases when the value η increases. P_{nor} and RN also increase, since the vacation realizes speedily when η increases.

Table 12 indicates the effect of p on various performance measures and the cost function. When the p value increases, the number of customers who leave the system after availing of the first part of the service increases. So the values of ECB, EC, and P_{normal} decrease. But the values of P_{idle}, P_{vac}, and RN decrease.

8. Optimal N

To find optimal N, we consider the following cost function.

$$\begin{aligned} EC &= k\theta \times CV \times P_{vac} + [k\theta + (n-k)\phi] \times CN \times P_{normal} \\ &\quad + ECQ \times HCQ + CSN \times RN + HCB \times ECB. \end{aligned}$$

8.1. MAP with positive correlation (MPC)

Table 13. Fix $n = 5, k = 2, m = 3$, $\theta = 14, \phi = 15, \eta = 5, p = 0.1$

N	ECQ	ECB	Pnor	Pvac	RN	EC
2	34.2296	0.6477	0.6811	0.3189	0.2495	758.6822
3	34.2276	0.6510	0.6807	0.3193	0.1906	746.7418
4	34.2249	0.6529	0.6806	0.3194	0.1831	745.1509
5	34.2231	0.6541	0.6805	0.3195	0.1811	744.6837
6	34.2222	0.6549	0.6804	0.3196	0.1801	744.4759
7	34.2216	0.6553	0.6804	0.3196	0.1797	744.3742
8	34.2213	0.6556	0.6804	0.3196	0.1795	744.3236
9	34.2211	0.6557	0.6804	0.3196	0.1794	744.2984
10	34.2211	0.6558	0.6804	0.3196	0.1793	744.2858
11	34.2210	0.6558	0.6804	0.3196	0.1793	744.2795
12	34.2210	0.6558	0.6804	0.3196	0.1793	744.2764
13	34.2210	0.6558	0.6804	0.3196	0.1793	744.2748
14	34.2210	0.6558	0.6804	0.3196	0.1793	744.2741
15	34.2210	0.6558	0.6804	0.3196	0.1793	744.2737
16	34.2210	0.6558	0.6804	0.3196	0.1793	744.2735
17	34.2210	0.6558	0.6804	0.3196	0.1793	**744.2734**
18	34.2210	0.6558	0.6804	0.3196	0.1793	744.2734
19	34.2210	0.6558	0.6804	0.3196	0.1793	744.2734
20	34.2210	0.6558	0.6804	0.3196	0.1793	744.2734

From Table 13, we get the expected cost corresponding to different values of N when the arrival process is MPC. We fix $n = 5, k = 2, m = 3$, $\theta = 14, \phi = 15, \eta = 5$. In this case, the minimum cost is 744.2734, obtained at $N = 17$. Therefore the optimal value of N is 17. After that cost remains constant because the vacation realization will happen.

8.2. MAP with negative correlation (MNC)

From Table 14, we get the expected cost corresponding to different values of N when the arrival process is MNC. We fix $n = 5, k = 2, m = 3$, $\theta = 14, \phi = 15, \eta = 5$. In this case, the minimum cost is 601.9418, obtained at $N = 8$. Therefore the optimal value of N is eight; after that cost remains constant because the vacation realization will happen.

8.3. MAP with zero correlation (MZC)

From Table 15, we get the expected cost corresponding to different values of N when the arrival process is MZC. We fix $n = 5, k = 2, m = 3$, $\theta = 14, \phi = 15, \eta = 5$. In this case, the minimum cost is 622.2760, obtained at $N = 14$. Therefore the optimal value of N is 14. After that cost remains constant because the vacation realization will happen.

Table 14. Fix $n = 5, k = 2, m = 3, \theta = 14, \phi = 15, \eta = 5, p = 0.1$

N	ECQ	ECB	Pnor	Pvac	RN	EC
2	1.3432	0.5836	0.5936	0.4064	0.4195	616.5412
3	1.3440	0.5862	0.5931	0.4069	0.3528	602.9791
4	1.3438	0.5863	0.5930	0.4070	0.3485	602.0623
5	1.3438	0.5863	0.5930	0.4070	0.3479	601.9529
6	1.3438	0.5863	0.5930	0.4070	0.3479	601.9430
7	1.3438	0.5863	0.5930	0.4070	0.3479	601.9419
8	1.3438	0.5863	0.5930	0.4070	0.3479	**601.9418**
9	1.3438	0.5863	0.5930	0.4070	0.3479	601.9418
10	1.3438	0.5863	0.5930	0.4070	0.3479	601.9418
11	1.3438	0.5863	0.5930	0.4070	0.3479	601.9418
12	1.3438	0.5863	0.5930	0.4070	0.3479	601.9418
15	1.3438	0.5863	0.5930	0.4070	0.3479	601.9418
20	1.3438	0.5863	0.5930	0.4070	0.3479	601.9418

Table 15. Fix $n = 5, k = 2, m = 3, \theta = 14, \phi = 15, \eta = 5, p = 0.1$

N	ECQ	ECB	Pnor	Pvac	RN	EC
2	1.5805	0.6246	0.6604	0.3396	0.5186	671.3343
3	1.5872	0.6363	0.6590	0.3410	0.3352	634.0300
4	1.5893	0.6406	0.6586	0.3414	0.2922	625.2544
5	1.5899	0.6421	0.6585	0.3415	0.2814	623.0515
6	1.5901	0.6426	0.6585	0.3415	0.2786	622.4813
7	1.5902	0.6428	0.6585	0.3415	0.2778	622.3310
8	1.5902	0.6428	0.6585	0.3419	0.2776	622.2909
9	1.5902	0.6428	0.6585	0.3419	0.2776	622.2801
10	1.5902	0.6428	0.6585	0.3419	0.2776	622.2771
11	1.5902	0.6428	0.6585	0.3419	0.2776	622.2763
12	1.5902	0.6428	0.6585	0.3419	0.2776	622.2761
13	1.5902	0.6428	0.6585	0.3419	0.2776	622.2761
14	1.5902	0.6428	0.6585	0.3419	0.2776	**622.2760**
15	1.5902	0.6428	0.6585	0.3419	0.2776	622.2760
16	1.5902	0.6428	0.6585	0.3419	0.2776	622.2760
17	1.5902	0.6428	0.6585	0.3419	0.2776	622.2760
18	1.5902	0.6428	0.6585	0.3419	0.2776	622.2760
19	1.5902	0.6428	0.6585	0.3419	0.2776	622.2760
20	1.5902	0.6428	0.6585	0.3419	0.2776	622.2760

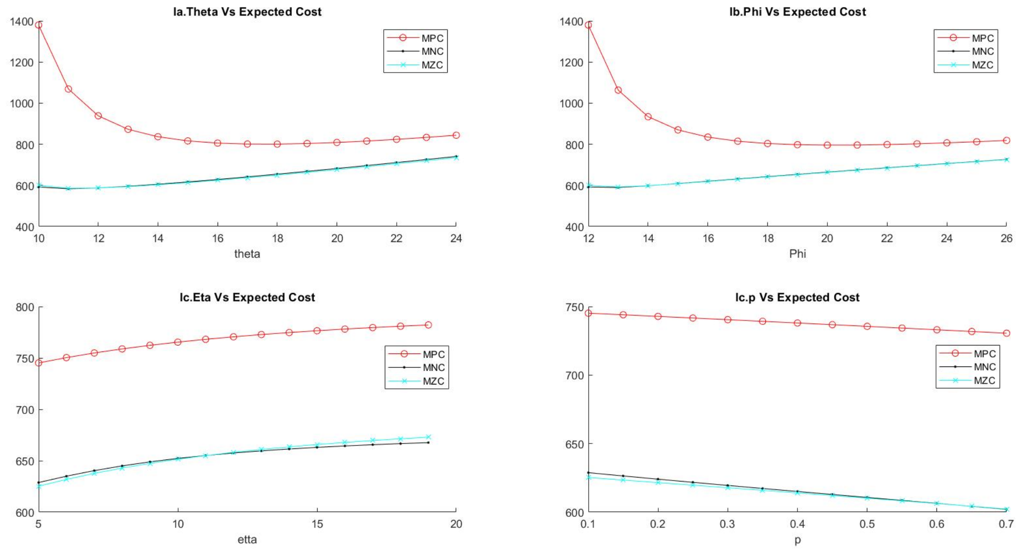

Figure. 1. Effect of θ, ϕ, η and p on Expected Cost

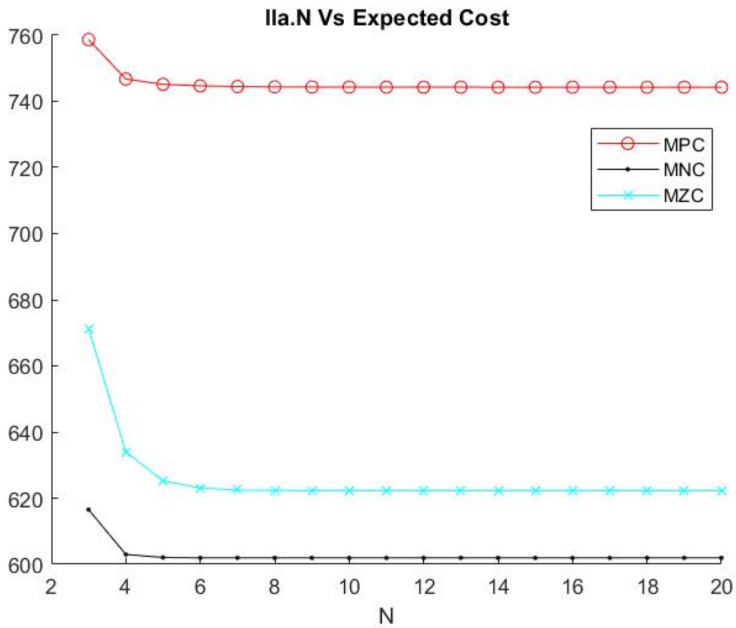

Figure. 2. Effect of N on Expected Cost

9. Conclusion

In this paper, we considered a $MAP/E_k/1$ queue with working vacation and N-Policy. During the working vacation, the server provides only the preliminary service. After availing of the preliminary service, a customer leaves the system with probability p and those who require the main service join a buffer of finite capacity N with complementary probability $1-p$. We analysed this model by using the matrix-analytic method. Several system performance characteristics were computed. Also, we constructed a cost function to find optimal N. Finally, we performed some numerical experiments to evaluate some performance measures and found optimal cost function values. We obtained the optimal values of N using the cost function for the Markovian arrival process's positive, zero and negative correlation values.

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