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On Steady State Analysis of an Infinite Capacity $M^X/G^{(a,Y)}/1$ Queue with Optional Service and Queue Length Dependent Single (Multiple) Vacation

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Abstract: This paper analyzes an infinite buffer bulk arrival batch size dependent bulk service queue with server's vacation. Customers enter the system in groups of random size following Poisson manner. Single server provides two kinds of services, first one is essential for all the joining customers and is termed as first essential service (FES) and the second service is optional to the customers and is termed as second optional service (SOS). The server serves the customers in batches following versatile bulk service (VBS) rule in FES. At the end of FES, the entire batch of customers served in FES will either join SOS with certain probability or leave the system. While server is providing SOS, the FES can not be resumed. The service time distribution for FES and SOS both are generally distributed while FES time distribution is considered to be batch size dependent. At the end of a round of service (which includes FES and/or SOS), upon looking into the queue length, the server will decide to go for vacation (SV or MV) with general vacation time distribution which depends on the queue length at vacation initiation epoch. We analyze the model mathematically by using the supplementary variable technique (SVT), embedded Markov chain technique, and bivariate generating function technique. We obtained the joint probabilities of the queue length and server content at the service completion (arbitrary) epoch. We also obtained the joint probabilities of the queue length and vacation type at the vacation completion (arbitrary) epoch. Several qualitative performance measures are obtained. Finally, Numerical results are also presented to see the behavior of the considered model.

Keywords: VBS rule, Second optional service (SOS), Single vacation (SV), Multiple vacation (MV), Bivariate generating function.

1. Introduction

In real life scenario, queueing models are analyzed to reduce the congestion because of their applicability in various areas, *viz.*, computer networking, traffic signal point, manufacturing point, transportation, etc. Customers enter the system either singly or in bulk in such queueing models. The server serves the customers either individually or in batch with

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different service rules, *viz*., fixed batch size service rule, general bulk service (GBS) rule, random batch size bulk service rule, versatile bulk service (VBS) rule, etc. For detail and deeper knowledge of bulk service queue, we mention Chaudhry and Templeton [12], and Medhi [28]. VBS rule, also called the (a, Y) rule is the most generalized bulk service rule among all other mentioned bulk service rule. It was proposed by Powell and Humblet [31], and later Kim et al. [21] named this rule as VBS rule. According to the VBS rule, the server provides the service in batches, and a is the minimum threshold limit to serve. After one service, if the queue length is found to be less than a, then the server remains in the idle state and waits the queue length to reach to a or more for starting service. At the beginning of the service the server takes $Y(= i), i = a, a+1, ..., B$ customers for the service, where B is the maximum capacity of the server, and Y is the random variable having finite support (Y is also called variable service capacity) with probability mass function (PMF) Pr($Y = i$)=y_i, $i = a, a+1, ..., B$ and $y_B > 0$. At the beginning of the service, if the number of customers is greater than or equal to a but less than the chosen service capacity i ($a \leq i \leq B$) then it does not wait for the queue length to reach i, and serves all the the customers with probability y_i. At the beginning of the service, if the number of customers is greater than the chosen service capacity i then it takes i customers for the service with probability y_i and rest will wait in the queue. Literature in which the authors considered the VBS rule can be found in [40, 25, 8, 33] and the references therein. If $y_B = 1$ then VBS rule converts in GBS rule which is proposed by Neuts [30].

Queueing system when no jobs are available in the system, the server goes into an idle state. During this time, the manager may have some additional work for the server, such as maintenance of the machine, promoting the company's new policy, etc. Hence, for utilizing the idle period of the server, Levy and Yechiali [23] proposed $M/G/1$ vacation queueing model with single vacation (SV) and multiple vacation (MV) in which the server performs some supplementary work when no primary job is available in the system. After this significant contribution in the literature, many researchers turned their attention towards the vacation model, see the excellent survey paper of Doshi [16], and Ke et al. [20]. For the quality literature on vacation theory, readers are requested to see Takagi [36], and Tian and Zhang [38]. According to the SV policy, after one service, if there is no job available for service, then the server goes down for vacation, and after the vacation completion, if there is a job in the system, then it starts service, otherwise, it remains in the dormant state. However, in the case of MV, it takes a repeated number of vacations until it finds the job at the end of the vacation completion.

The terminology of second optional service in queueing systems was proposed by Madan [24], where he analyzed the $M/G/1$ queueing model using SVT. For every customer, the first service is essential and is called first essential service (FES), some of them request the subsidiary service called second optional service (SOS). Later, many researchers have analyzed different queueing models with a second optional service, see, Medhi [27], Al-Jararha and Madan [3], Wang [39], Choudhury and Tadj [11].

1.1. Related literature survey

The literature on bulk service queues with the VBS rule can be found in [40, 8] where authors obtained the queue length distribution at various epochs for different queues, however, few authors worked for the joint distribution of queue and server content, see [25, 26, 33]. Maity and Gupta [25] analyzed $M/M^{(a,Y)}/1$ queue and obtained the joint distribution of queue and server content in steady-state. Maity et al. [26] considered $Geo/G_n^{(a,Y)}/1$ queue and obtained the joint distribution of queue and server content at various epochs. Recently, Pradhan [33] analyzed $M/G_n^{(a,Y)}/1$ queue and obtained the joint distribution of queue and server content at various epochs by using the supplementary variable technique and the bivariate probability generating function (PGF) method.

The literature on bulk service queueing models with vacation where the customers gets the service according to the GBS rule are analyzed in [17, 29, 37] and the references therein. Tamrakar and Banerjee [37] considered $M/G_r^{(a,b)}/1$ queue with queue length dependent single and multiple vacation. They obtained joint probabilities of queue and server content and the joint probabilities of queue length and vacation type at various epochs using SVT.

The bulk arrival bulk service queue with server's vacation has been analyzed by [10, 34, 19, 18, 4]. Chang and Choi [10] considered discrete time $Geo^X/G^Y/1/N$ queue with MV and obtained some performance measures at various epochs. Sikdar and Gupta [34] considered $M^X/G^Y/1/N$ queue with SV and MV and obtained the stationary queue length distribution at various epochs by using SVT. Haridass and Arumuganathan [19] analyzed $M^X/G^{(a,b)}/1$ queue with vacation interruption and obtained queue length distribution at an arbitrary epoch. Jeyakumar and Senthilnathan [18] considered $M^X/G^{(a,b)}/1$ with MV and setup time, closedown times, and server breakdown without interruption. Using SVT, they derived the PGF of the queue length at an arbitrary epoch. Ayyappan and Deepa [4] considered $M^X/G^{(a,b)}/1$ queue with MV, closedown, essential and optional repair, and obtained the queue size distribution at an arbitrary epoch using the SVT.

For the bulk queues with SOS, few papers are available in the literature . Ayyappan and Shyamala [6] considered $M^X/G/1$ queue with SOS, Bernoulli Schedule server vacation and random break downs and obtained the time-dependent probability generating functions in terms of the Laplace transforms and the corresponding steady state results are also obtained explicitly. Ayyappan and Supraja [7] analyzed $M^X/G^{(a,b)}/1$ queue with unreliable server, second optional service, two different vacations, and restricted admissibility policy, and obtained the queue length distribution at random and departure epoch using the SVT. Singh et al. [35] analyzed bulk arrival queue with different m-SOS, vacation, and unreliable server using SVT. Ayyappan and Deepa [5] considered $M^X/G^{(a,b)}/1$ queue with SOS, MV, and setup time. They obtained the PGF of the queue size at different epochs using SVT. For the current work on the bulk queues with SOS, readers are invoked to see [22, 15] and the references therein. To the best of the authors' knowledge, the considered model, i.e., $M^X/G^{(a,Y)}/1$ queue with SOS and queue length dependent SV and MV, has not been analyzed so far in the literature for the joint probabilities of queue and server content for FES (SOS) at the service completion (arbitrary) epoch as well as joint probabilities of queue content and vacation type at the vacation termination (arbitrary) epoch.

1.2. Practical motivation

Queueing models with SOS can be applied in many areas (*viz*., Barbar shop, Malls, etc.) Considered model can be used for blood (or swab) sample testing in an epidemic situation such as COVID-19, as batch service queues have efficacious application in blood pooling, see, e.g., [2, 9, 14]. In an epidemic (*viz*., COVID-19), the health administration of any country wants to test more and more samples using less number of kits. Hence, a mixed sample is used for testing by taking a group of samples from the queue, see [41, 32, 13]. Further, in a pandemic situation handling the health workers' shortage is also a big challenge. To deal with such situation, the health administration may provide some additional work to the health workers (*viz*., visiting the quarantine room, stocking the health care inventory, making people aware of the epidemic) when he has no primary work.

Our model may play a key role in sample testing to deal with pandemic situations such as COVID-19. Let us consider that a large number of samples arrive at the health department in bulk for testing from different sectors, then the health worker tests these samples in batches (mixed sample), termed as FES, according to the (a, Y) rule with batch size dependent service. After FES, if the mixed sample diagnosed negative then the health worker decides how many samples will be mixed for the next test with a certain probability. For example, the health administration instructs the health worker that if the mixed sample is found negative, select the batch of maximum capacity for the next test; otherwise, choose the batch size of its minimum capacity. Therefore, (a, Y) rule is justified here. After FES, if the mixed sample diagnosed positive then the sample go for the SOS to identify the infected sample.

Further, in the absence of primary work, the health worker does some additional work (*viz*., stocking of health care inventory, increase people awareness, visiting the quarantine room, etc.). Before going for this additional work, the health worker always checks the queue size, and depending on the queue size; he fixes his returning time in the primary system. Hence, the QSDV policy rule may have a wide impact on the system's performance. The practical application discusses above motivate us to work on this problem.

1.3. Paper structure

Section 2 presents the model description of the considered model. In Section 3, joint probabilities of queue and server content as well as queue length and vacation type obtained at different epoch. The various important performance measures are presented in Section 4. The behavior of the system is discussed by means of graphs and tables in Section 5. The whole study ends with the conclusion (i.e., Section 6).

2. Model

The present paper investigates infinite capacity bulk arrival, batch size dependent bulk service queue with queue size dependent single (multiple) vacations. Here below is the detail mathematical description of our model.

The customers are coming in packets (groups) following the Poison distribution with rate λ. Let X be the size of the arriving group with probability mass function $Pr(X = m) = g_m$, $m \in \mathbb{N}$ associated with finite mean $E(X) = \tilde{g}$ and PGF $X(z) = \sum_{i=1}^{\infty} g_i z^i$. The customers are served in batches according to the VBS rule, i.e., (a, Y) rule, where the random variable Y, denoting service capacity, has the following probability mass function,

$$Pr(Y = i) = \begin{cases} y_i, & a \leq i \leq B \\ 0, & otherwise. \end{cases}$$

Here B is the maximum serving capacity of the server with $y_B > 0$ and $E(Y) = \tilde{y}$. At each service initiation epoch if the queue length lies in $[a, i)$ (where i is the chosen service capacity at the service initiation epoch) then server does not wait for the queue length to reach i, but it takes entire customer for the service with probability y_i, and if the server finds the queue length $\geq i$ then it takes only i customers for the service with probability y_i. The service (FES) time (T_r), of a batch of size r $(a \leq r \leq B)$ is distributed generally along with probability density function (pdf) $s_r(t)$, distribution function (DF) $S_r(t)$, the Laplace-Stieltjes transform (LST) $\tilde{S}_r(\theta)$ and the mean service time $\frac{1}{\mu_r} = s_r = -\tilde{S}_r^{(1)}(0)$ $(a \leq r \leq B)$, where $\tilde{S}_r^{(1)}(0)$ is the derivative of $\tilde{S}_r(\theta)$ evaluated at $\theta = 0$. After first essential service (FES) the served batch may choose second optional service (SOS) with probability α. The optional service time $(\hat{T})$ of a batch distributed generally along with probability density function (pdf) $s(t)$, distribution function (DF) $S(t)$, the Laplace-Stieltjes transform (LST) $\tilde{S}(\theta)$ and the mean service time $\frac{1}{\mu} = \varsigma = -\tilde{S}^{(1)}(0)$, where $\tilde{S}^{(1)}(0)$ is the derivative of $\tilde{S}(\theta)$ evaluated at $\theta = 0$. After FES if the queue length is found to be less than the minimum threshold limit a and the batch served in FES does not choose SOS then the server goes for the type k vacation (where k $(0 \leq k \leq a-1)$ is the queue length at vacation initiation epoch), similarly, after SOS if the queue length is found to be $k < a$ then the server goes for type k vacation . At the end of the vacation if the queue length is $\geq a$ then it serves the customer as per the (a, Y) rule otherwise depending on the vacation policy the server remains in the system at dormant state until queue length reaches at least the minimum threshold limit a or takes repeated vacation until it finds queue length $\geq a$ at the end of the vacation. Vacation time V_k of the type k vacation obeys the general distribution with pdf $v_k(t)$, DF $V_k(t)$, LST $\tilde{V}_k(\theta)$. The mean vacation time $\frac{1}{\nu_k} = x_k = -\tilde{V}_k^{(1)}(0)$ where $\tilde{V}_k^{(1)}(0)$ is the derivative of $\tilde{V}_k(\theta)$ at $\theta = 0$.

The traffic intensity of the system $\rho = \frac{\lambda\tilde{g}\sum_{i=a}^{B}\frac{y_i}{\mu_i} + \lambda\tilde{g}\frac{\alpha}{\mu}}{\tilde{y}} (< 1)$ which ensures the stability of the system. In this paper we have studied SV and MV queues in an unified way by defining a variable δ as follows:

$$\delta = \begin{cases} 1, & \text{for MV}, \\ 0, & \text{for SV}. \end{cases}$$

Remark 1: The traffic intensity of the model under consideration is proved using the result given in Abolnikov and Dukhovny [1]. If we consider the TPM of the queue length at complete round of service completion epoch (i.e., FES and/or SOS). Then the resulting TPM will be $\Delta_{B,B}$ type matrix. Then using the stability condition given in Abolnikov and Dukhovny [[1], Theorem 3.4], the Markov chain is ergodic if and only if $\left\{\frac{d}{dz}(1-\alpha+\alpha M_{os}(z))\sum_{i=a}^{B} y_i M^{(i)}(z) z^{B-i}\right\}_{z=1} < B$, i.e., $\rho < 1$, where $M^{(r)}(z)$ $(a \le r \le B)$ and $M_{os}(z)$ are the PGF of $m_j^{(r)}$ and q_j, respectively, with $m_j^{(r)}$ = $Pr\{j$ arrivals during the service (i.e., FES) time of a batch size $r\}$, $a \le r \le B$, $j \ge 0$, and q_j = $Pr\{j$ arrivals during the service (i.e., SOS) time$\}$, $j \ge 0$.

3. Analysis

This section is devoted in obtaining the joint probabilities of the queue length and server content at the service (FES and SOS) completion epoch and the joint probabilities of the queue size and the vacation type at the vacation termination epoch. Later we obtain the joint probabilities of the queue and server content during FES (SOS) and the joint probabilities of queue length and vacation type at an arbitrary epoch by relating it to the joint probabilities at service completion and vacation termination epoch. From this perspective, we have the following random variable at time t.

- $N(t)$: be the number of customers are in the queue.
- $\hat{S}_1(t)$: be the number of customers with the server when the server is busy in FES.
- $\hat{S}_2(t)$: be the number of customers with the server when the server is busy in SOS.
- $\hat{K}(t)$: be the vacation type taken by the server, when the server is on vacation.
- $U(t)$: remaining service (FES) time of the batch, if any.
- $\hat{U}(t)$: remaining service (SOS) time of the batch, if any.
- $V(t)$: remaining vacation time of the server, if any.

Point to be noted here that $\hat{S}_1(t) = 0$ and $\hat{S}_2(t) = 0$ will represent the server is in the dormant state at time t.

For SV, $\{(N(t), \hat{S}_1(t) = 0, \hat{S}_2(t) = 0), 0 \le n \le a-1\} \cup \{(N(t), \hat{S}_1(t), U(t))\} \cup \{(N(t), \hat{S}_2(t), \hat{U}(t))\} \cup \{(N(t), \hat{K}(t), V(t))\}$ forms a Markov chain with state space $\{(n,0,0);\ 0 \le n \le a-1\} \bigcup \{(n,r,u); n \ge 0, a \le r \le B, u \ge 0\} \bigcup \{(n,k,u); 0 \le k \le a-1, n \ge k, u \ge 0\}$.

For MV, $\{(N(t), \hat{S}_1(t), U(t))\} \cup \{(N(t), \hat{S}_2(t), \hat{U}(t))\} \cup \{(N(t), \hat{K}(t), V(t))\}$ forms a Markov chain with state space $\{(n,r,u); n \ge 0, a \le r \le B, u \ge 0\} \bigcup \{(n,k,u); 0 \le k \le a-1, n \ge k, u \ge 0\}$.

Let us now, define the state probabilities at time t as

- $R_n(t) \equiv Pr\{N(t) = n, \hat{S}_1(t) = 0, \hat{S}_2(t) = 0\}$, $0 \le n \le a-1$ (exist only for SV).
- $P_{n,r}(u,t)du \equiv Pr\{N(t) = n, \hat{S}_1(t) = r, u \le U(t) \le u + du\}$, $n \ge 0$, $a \le r \le B$.
- $W_{n,r}(u,t)du \equiv Pr\{N(t) = n, \hat{S}_2(t) = r, u \le \hat{U}(t) \le u + du\}$, $n \ge 0$, $a \le r \le B$.

- $Q_n^{[k]}(u,t)du \equiv \; Pr\{N(t)=n, \hat{K}(t)=k, u \leq V(t) \leq u+du\}, \;\; n \geq k \;,\; 0 \leq k \leq a-1.$

In steady state, as $t \to \infty$, the limiting probabilities are defined as follows

$R_n = \lim_{t\to\infty} R_n(t)$ $(0 \leq n \leq a-1)$, (exist only for SV),

$P_{n,r}(u) = \lim_{t\to\infty} P_{n,r}(u,t), \;\; n \geq 0, \;\; a \leq r \leq B,$

$W_{n,r}(u) = \lim_{t\to\infty} W_{n,r}(u,t), \;\; n \geq 0, \;\; a \leq r \leq B,$

$Q_n^{[k]}(u) = \lim_{t\to\infty} Q_n^{[k]}(u,t), \;\; n \geq k, \;\; 0 \leq k \leq a-1.$

Now we obtain the system equation that governs the system behavior. Analyzing the system, at time t and $t+dt$, in steady state, the Kolmogorov equations are obtained as follows:

$$0 = (1-\delta)\Big(-\lambda R_0 + Q_0^{[0]}(0)\Big), \tag{1}$$

$$0 = (1-\delta)\Big(-\lambda R_n + \lambda\sum_{i=1}^{n} g_i R_{n-i} + \sum_{k=0}^{n} Q_n^{[k]}(0)\Big), \;\; 1 \leq n \leq a-1, \tag{2}$$

$$\begin{aligned} -\frac{d}{du}P_{0,r}(u) &= -\lambda P_{0,r}(u) \\ &+\Big(\sum_{k=0}^{a-1} Q_r^{[k]}(0) + \sum_{j=a}^{B} P_{r,j}(0)(1-\alpha) + \sum_{j=a}^{B} W_{r,j}(0)\Big)\sum_{i=r}^{B} y_i s_r(u) \\ &+(1-\delta)\lambda\sum_{j=0}^{a-1} R_j g_{r-j}\sum_{i=r}^{B} y_i s_r(u), \;\; a \leq r \leq B, \end{aligned} \tag{3}$$

$$\begin{aligned} -\frac{d}{du}P_{n,r}(u) &= -\lambda P_{n,r}(u) + \lambda\sum_{j=1}^{n} P_{n-j,r}(u) g_j \\ &+\Big(\sum_{k=0}^{a-1} Q_{n+r}^{[k]}(0) + \sum_{j=a}^{B} P_{n+r,j}(0)(1-\alpha) + \sum_{j=a}^{B} W_{n+r,j}(0)\Big) y_r s_r(u) \\ &+(1-\delta)\lambda\sum_{j=0}^{a-1} R_j g_{n+r-j} y_r s_r(u), \; a \leq r \leq B, \; n \geq 1, \end{aligned} \tag{4}$$

$$\begin{aligned} -\frac{d}{du}Q_k^{[k]}(u) &= -\lambda Q_k^{[k]}(u) + \Big(\sum_{r=a}^{B} P_{k,r}(0)(1-\alpha) + \sum_{r=a}^{B} W_{k,r}(0) \\ &+\delta\sum_{j=0}^{k} Q_k^{[j]}(0)\Big) v_k(u), \;\; 0 \leq k \leq a-1, \end{aligned} \tag{5}$$

$$-\frac{d}{du}Q_n^{[k]}(u) = -\lambda Q_n^{[k]}(u) + \lambda\sum_{i=1}^{n-k} g_i Q_{n-i}^{[k]}(u), \;\; n \geq k+1, \;\; 0 \leq k \leq a-1, \tag{6}$$

$$-\frac{d}{du}W_{0,r}(u) = -\lambda W_{0,r}(u) + P_{0,r}(0)s(u)\alpha, \;\; a \le r \le B, \tag{7}$$

$$-\frac{d}{du}W_{n,r}(u) = -\lambda W_{n,r}(u) + \lambda \sum_{j=1}^{n} W_{n-j,r}(u)g_j + P_{n,r}(0)s(u)\alpha,$$
$$n \ge 1, a \le r \le B. \tag{8}$$

Further, we define for Re $\theta \ge 0$,

$$\tilde{S}_r(\theta) = \int_0^\infty e^{-\theta u} dS_r(u) = \int_0^\infty e^{-\theta u} s_r(u)du, \;\; a \le r \le B, \tag{9}$$

$$\tilde{P}_{n,r}(\theta) = \int_0^\infty e^{-\theta u} P_{n,r}(u)du, \;\; a \le r \le B, \; n \ge 0, \tag{10}$$

$$P_{n,r} \equiv \tilde{P}_{n,r}(0) = \int_0^\infty P_{n,r}(u)du, \;\; a \le r \le B, \; n \ge 0, \tag{11}$$

$$\tilde{S}(\theta) = \int_0^\infty e^{-\theta u} dS(u) = \int_0^\infty e^{-\theta u} s(u)du, \;\; a \le r \le B, \tag{12}$$

$$\tilde{W}_{n,r}(\theta) = \int_0^\infty e^{-\theta u} W_{n,r}(u)du, \;\; a \le r \le B, \; n \ge 0, \tag{13}$$

$$W_{n,r} \equiv \tilde{W}_{n,r}(0) = \int_0^\infty W_{n,r}(u)du, \;\; a \le r \le B, \; n \ge 0, \tag{14}$$

$$\tilde{V}_k(\theta) = \int_0^\infty e^{-\theta u} dV_k(u) = \int_0^\infty e^{-\theta u} v_k(u)du, \;\; 0 \le k \le a-1, \tag{15}$$

$$\tilde{Q}_n^{[k]}(\theta) = \int_0^\infty e^{-\theta u} Q_n^{[k]}(u)du, \;\; 0 \le k \le a-1, \; n \ge k, \tag{16}$$

$$Q_n^{[k]} \equiv \tilde{Q}_n^{[k]}(0) = \int_0^\infty Q_n^{[k]}(u)du, \;\; 0 \le k \le a-1, \;\; n \ge k. \tag{17}$$

One may note here that $P_{n,r}$ ($W_{n,r}$) denotes the probability that there are n ($n \ge 0$) customers in the queue and server is busy with r ($a \le r \le B$) customers during FES (SOS), at an arbitrary epoch. Also, $Q_n^{[k]}$ indicates the probability of n ($n \ge k$) customers in the queue and the server is on type k ($0 \le k \le a-1$) vacation, at an arbitrary epoch. Multiplying (3)-(8) by $e^{-\theta u}$ and integrating with respect to u over 0 to ∞ we obtain

$$(\lambda - \theta)\tilde{P}_{0,r}(\theta) = \left(\sum_{k=0}^{a-1} Q_r^{[k]}(0) + \sum_{j=a}^{B} P_{r,j}(0)(1-\alpha) + \sum_{j=a}^{B} W_{r,j}(0) \right) \sum_{i=r}^{B} y_i \tilde{S}_r(\theta)$$
$$+(1-\delta)\lambda \sum_{j=0}^{a-1} R_j g_{r-j} \sum_{i=r}^{B} y_i \tilde{S}_r(\theta) - P_{0,r}(0), \;\; a \le r \le B, \tag{18}$$

$$(\lambda - \theta)\tilde{P}_{n,r}(\theta) = \lambda \sum_{j=1}^{n} g_j \tilde{P}_{n-j,r}(\theta) + \Bigg(\sum_{k=0}^{a-1} Q_{n+r}^{[k]}(0) + \sum_{j=a}^{B} P_{n+r,j}(0)(1-\alpha)$$

$$
\begin{aligned}
& + \sum_{j=a}^{B} W_{n+r,j}(0)\Big) y_r \tilde{S}_r(\theta) + (1-\delta)\lambda \sum_{j=0}^{a-1} R_j g_{n+r-j} y_r \tilde{S}_r(\theta) \\
& - P_{n,r}(0), \ a \le r \le B, \ n \ge 1, \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
(\lambda-\theta)\tilde{Q}_k^{[k]}(\theta) &= \Big(\sum_{r=a}^{B} P_{k,r}(0)(1-\alpha) + \sum_{r=a}^{B} W_{k,r}(0) + \delta \sum_{j=0}^{k} Q_k^{[j]}(0) \Big) \tilde{V}_k(\theta) \\
& \quad - Q_k^{[k]}(0), \ \ 0 \le k \le a-1, \qquad (20) \\
(\lambda-\theta)\tilde{Q}_n^{[k]}(\theta) &= \lambda \sum_{i=1}^{n-k} g_i \tilde{Q}_{n-i}^{[k]}(\theta) - Q_n^{[k]}(0), \ n \ge k+1, \ \ 0 \le k \le a-1, \qquad (21) \\
(\lambda-\theta)\tilde{W}_{0,r}(\theta) &= P_{0,r}(0)\tilde{S}(\theta)\alpha - W_{0,r}(0), \ \ a \le r \le B, \qquad (22) \\
(\lambda-\theta)\tilde{W}_{n,r}(\theta) &= \lambda \sum_{j=1}^{n} \tilde{W}_{n-j,r}(\theta) g_j + P_{n,r}(0)\tilde{S}(\theta)\alpha \\
& \quad - W_{n,r}(0), \ \ n \ge 1, a \le r \le B. \qquad (23)
\end{aligned}
$$

As our main objective is to obtain the joint probabilities of the queue and server content during FES (SOS) as well as the joint probabilities of queue length and vacation type at an arbitrary epoch, these arbitrary epoch joint probabilities are obtained by establishing a relationship between the joint probabilities of the queue length and server content at the service completion epoch, and the joint probabilities of the queue length and vacation type at the vacation termination epoch. Towards this end, we define,

$$
\begin{aligned}
P_{n,r}^{+} &= Pr\{n \text{ customers are in the queue at service (i.e., FES) completion epoch} \\
& \qquad \text{of a batch of size } r\}, \ \ n \ge 0, \ \ a \le r \le B, \qquad (24) \\
P_n^{+} &= Pr\{n \text{ customers are in the queue at service (i.e., FES) completion epoch}\} \\
&= \sum_{r=a}^{B} P_{n,r}^{+}, \ \ n \ge 0, \qquad (25) \\
W_{n,r}^{+} &= Pr\{n \text{ customers are in the queue at service (i.e., SOS) completion epoch} \\
& \qquad \text{of a batch of size } r\}, \ \ n \ge 0, \ \ a \le r \le B, \qquad (26) \\
W_n^{+} &= Pr\{n \text{ customers are in the queue at service (i.e., SOS) completion epoch}\} \\
&= \sum_{r=a}^{B} W_{n,r}^{+}, \ \ n \ge 0, \qquad (27) \\
Q_n^{[k]+} &= Pr\{n \text{ customers are in the queue at type } k \text{ vacation} \\
& \quad \text{termination epoch}\}, \ \ 0 \le k \le a-1, \qquad (28) \\
Q_n^{+} &= Pr\{n \text{ customers are in the queue at the vacation termination epoch}\} \\
&= \sum_{k=0}^{min(n,a-1)} Q_n^{[k]+}, \ \ n \ge 0. \qquad (29)
\end{aligned}
$$

3.1. Joint probabilities at service (vacation) completion epoch

In this subsection our primary objective is to obtain $P^+_{n,r}$ $(W^+_{n,r})$ $(n \geq 0, a \leq r \leq B)$ and $Q^{[k]+}_n$ $(0 \leq k \leq a-1, n \geq k)$, i.e., the joint probabilities of the queue length and server content at service (FES and SOS) completion epoch and the joint probabilities of the queue size and the vacation type at vacation termination epoch, in this connection we further define the following generating functions:

$$P(z,y,\theta) = \sum_{n=0}^{\infty}\sum_{r=a}^{B} \tilde{P}_{n,r}(\theta) z^n y^r, |z| \leq 1, \ |y| \leq 1, \tag{30}$$

$$P^+(z,y) = \sum_{n=0}^{\infty}\sum_{r=a}^{B} P^+_{n,r} z^n y^r, |z| \leq 1, \ |y| \leq 1, \tag{31}$$

$$P^+(z,1) = \sum_{n=0}^{\infty}\sum_{r=a}^{B} P^+_{n,r} z^n = \sum_{n=0}^{\infty} P^+_n z^n = P^+(z), \ \ |z| \leq 1, \tag{32}$$

$$W(z,y,\theta) = \sum_{n=0}^{\infty}\sum_{r=a}^{B} \tilde{W}_{n,r}(\theta) z^n y^r, |z| \leq 1, \ |y| \leq 1, \tag{33}$$

$$W^+(z,y) = \sum_{n=0}^{\infty}\sum_{r=a}^{B} W^+_{n,r} z^n y^r, |z| \leq 1, \ |y| \leq 1, \tag{34}$$

$$W^+(z,1) = \sum_{n=0}^{\infty}\sum_{r=a}^{B} W^+_{n,r} z^n = \sum_{n=0}^{\infty} W^+_n z^n = W^+(z), \ \ |z| \leq 1, \tag{35}$$

$$Q(z,y,\theta) = \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} \tilde{Q}^{[k]}_n(\theta) z^n y^k, |z| \leq 1, \ |y| \leq 1, \tag{36}$$

$$Q^+(z,y) = \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q^{[k]+}_n z^n y^k, |z| \leq 1, \ |y| \leq 1, \tag{37}$$

$$\begin{aligned} Q^+(z,1) &= \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q^{[k]+}_n z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{min(n,a-1)} Q^{[k]+}_n z^n \\ &= \sum_{n=0}^{\infty} Q^+_n z^n = Q^+(z), \ \ |z| \leq 1. \end{aligned} \tag{38}$$

Further, we define

$$\begin{aligned} m^{(r)}_j &= Pr\{j \text{ arrivals during the service (i.e., FES) time of a batch size } r\}, \\ & \quad a \leq r \leq B, j \geq 0, \\ &= \begin{cases} \int_0^\infty \sum\limits_{l=1}^{j} \frac{e^{-\lambda t}(\lambda t)^l}{l!} g^{l(*)}_j s_r(t)dt, \ j \geq 1, \\ \int_0^\infty e^{-\lambda t} s_r(t)dt, \ j = 0. \end{cases} \end{aligned} \tag{39}$$

$$\begin{aligned} q_j &= Pr\{j \text{ arrivals during the service (i.e., SOS) time}\}, \;\; j \geq 0, \\ &= \begin{cases} \int_0^\infty \sum\limits_{l=1}^{j} \frac{e^{-\lambda t}(\lambda t)^l}{l!} g_j^{l(*)} s(t)dt, \; j \geq 1, \\ \int_0^\infty e^{-\lambda t} s(t)dt, \; j = 0. \end{cases} \end{aligned} \tag{40}$$

$$\begin{aligned} w_j^{(k)} &= Pr\{j \text{ arrivals during the type } k \text{ vacation}\}, \;\; 0 \leq k \leq a-1, j \geq 0, \\ &= \begin{cases} \int_0^\infty \sum\limits_{l=1}^{j} \frac{e^{-\lambda t}(\lambda t)^l}{l!} g_j^{l(*)} v_k(t)dt, \; j \geq 1, \\ \int_0^\infty e^{-\lambda t} v_k(t)dt, \; j = 0, \end{cases} \end{aligned} \tag{41}$$

where $g_j^{l(*)}$ is the probability associated with l-fold convolution function of g_j with itself. For detail discussion of the l-fold convolution associated with random variables readers are invoked to see the books by Chaudhry and Templeton [[12], Section 1.2]. Define the PGF (probability generating function) of $m_j^{(r)}$, q_j and $w_j^{(k)}$ are as follows:

$$M^{(r)}(z) = \sum_{j=0}^{\infty} m_j^{(r)} z^j = \tilde{S}_r(\lambda - \lambda X(z)), \;\; a \leq r \leq B, \;\; |z| \leq 1, \tag{42}$$

$$M_{os}(z) = \sum_{j=0}^{\infty} q_j z^j = \tilde{S}(\lambda - \lambda X(z)), \;\; a \leq r \leq B, \;\; |z| \leq 1, \tag{43}$$

$$N^{(k)}(z) = \sum_{j=0}^{\infty} w_j^{(k)} z^j = \tilde{V}_k(\lambda - \lambda X(z)), \;\; 0 \leq k \leq a-1, \;\; |z| \leq 1. \tag{44}$$

Lemma 1. *For the case of SV the following result is hold*

$$\lambda R_n = \sum_{m=0}^{n} \sum_{k=0}^{m} e_{n,m} Q_m^{[k]}(0), \;\; 0 \leq n \leq a-1. \tag{45}$$

where $e_{n,m} = \sum\limits_{i=1}^{n-m} g_i e_{n-i,m}, \;\; 1 \leq n \leq a-1, \;\; 0 \leq m \leq n-1$ *and* $e_{n,n} = 1, 0 \leq n \leq a-1.$

Proof. Setting $n = 1$ in (2) and using (1) we obtain

$$\lambda R_1 = \sum_{m=0}^{1} \sum_{k=0}^{m} e_{1,m} Q_m^{[k]}(0), \;\; e_{1,0} = g_1, e_{1,1} = 1. \tag{46}$$

Setting $n = 2$ in (2) and using (1) and (46) we obtain

$$\lambda R_2 = \sum_{m=0}^{2} \sum_{k=0}^{m} e_{2,m} Q_m^{[k]}(0), \;\; e_{2,0} = g_2 + e_{1,0} g_1, e_{2,1} = g_1, e_{2,2} = 1. \tag{47}$$

We repeat the above process by setting $n = 3, 4, ..., a-1$, respectively, in (2). In genral we get

$$\lambda R_n = \sum_{m=0}^{n}\sum_{k=0}^{m} e_{n,m} Q_m^{[k]}(0), \quad 1 \le n \le a-1, \tag{48}$$

where $e_{n,m} = \sum_{i=1}^{n-m} g_i e_{n-i,m}, \ 1 \le n \le a-1, \ 0 \le m \le n-1$ and $e_{n,n} = 1$, $1 \le n \le a-1$. Using (1), (48), and $e_{0,0} = 1$ we obtain the desire result (45).

Lemma 2. *The probabilities $P_{n,r}^{+}$, $W_{n,r}^{+}$, $Q_n^{[k]+}$, $P_{n,r}(0)$, $W_{n,r}(0)$ and $Q_n^{[k]}(0)$ $(a \le r \le B, 0 \le k \le a-1)$ are associated with the following relation*

$$P_{n,r}^{+} = \sigma P_{n,r}(0), \tag{49}$$
$$W_{n,r}^{+} = \sigma W_{n,r}(0), \tag{50}$$
$$Q_n^{[k]+} = \sigma Q_n^{[k]}(0), \tag{51}$$

where $\sigma^{-1} = \sum_{m=0}^{\infty}\sum_{r=a}^{B} P_{m,r}(0) + \sum_{m=0}^{\infty}\sum_{r=a}^{B} W_{m,r}(0) + \sum_{m=0}^{\infty}\sum_{k=0}^{min(m,a-1)} Q_m^{[k]}(0).$

Proof. Since $P_{n,r}^{+}$, $W_{n,r}^{+}$, and $Q_n^{[k]+}$ are proportional to $P_{n,r}(0)$, $W_{n,r}(0)$ and $Q_n^{[k]}(0)$, respectively. Applying the Bayes' theorem and $\sum_{n=0}^{\infty}\sum_{r=a}^{B}(P_{n,r}^{+} + W_{n,r}^{+}) + \sum_{n=0}^{\infty}\sum_{k=0}^{min(n,a-1)} Q_n^{[k]+}$=1 we get the desired outcome.

Lemma 3. *The value σ^{-1} is given by*

$$\sigma^{-1} = \frac{1-(1-\delta)\sum_{n=0}^{a-1} R_n}{f}, \tag{52}$$

where

$$\begin{aligned} f &= \sum_{n=B+1}^{\infty} \beta_n^{+} \sum_{r=a}^{B} y_r s_r + \beta_a^{+} s_a + \sum_{n=a+1}^{B} \beta_n^{+} \Big(\sum_{i=a}^{n-1} y_i s_i + \sum_{i=n}^{B} y_i s_n\Big) \\ &\quad + \sum_{n=0}^{a-1} \Big\{ P_n^{+} x_n (1-\alpha) + W_n^{+} x_n + (1-\delta) \sum_{m=n}^{a-1} e_{m,n} Q_n^{+} \Big(\sum_{r=a}^{B} g_{r-m} \sum_{i=r}^{B} y_i \mu_r \\ &\quad + \sum_{l=1}^{\infty}\sum_{r=a}^{B} g_{r-m+l} y_r \mu_r \Big) + \delta Q_n^{+} x_n \Big\} + \sum_{n=0}^{\infty} P_n^{+} \varsigma \alpha, \end{aligned}$$

and $\beta_n^{+} = P_n^{+}(1-\alpha) + W_n^{+} + Q_n^{+}$, $n \ge 0$.

Proof. Using (45), summing (18)-(23), we get

$$\sum_{m=0}^{\infty}\sum_{r=a}^{B}(\tilde{P}_{m,r}(\theta)+\tilde{W}_{m,r}(\theta))+\sum_{m=0}^{\infty}\sum_{k=0}^{min(m,a-1)}\tilde{Q}_m^{[k]}(\theta)=\frac{A(\theta)}{\theta}, \tag{53}$$

where

$$\begin{aligned}
A(\theta) \;=\; & \sum_{n=B+1}^{\infty}\left(\sum_{r=a}^{B}P_{n,r}(0)(1-\alpha)+\sum_{r=a}^{B}W_{n,r}(0)+\sum_{k=0}^{a-1}Q_n^{[k]}(0)\right)\sum_{r=a}^{B}y_r(1-\tilde{S}_r(\theta)) \\
& +\sum_{n=a+1}^{B}\left(\sum_{r=a}^{B}P_{n,r}(0)(1-\alpha)+\sum_{r=a}^{B}W_{n,r}(0)+\sum_{k=0}^{a-1}Q_n^{[k]}(0)\right) \\
& \left(\sum_{i=a}^{n-1}y_i(1-\tilde{S}_i(\theta))+\sum_{i=n}^{B}y_i(1-\tilde{S}_n(\theta))\right)+\sum_{n=0}^{a-1}\left(\sum_{r=a}^{B}P_{n,r}(0)(1-\alpha)\right. \\
& \left.+\sum_{r=a}^{B}W_{n,r}(0)+\delta\sum_{k=0}^{n}Q_n^{[k]}(0)\right)(1-\tilde{V}_n(\theta)) \\
& +\left(\sum_{j=a}^{B}P_{a,j}(0)(1-\alpha)+\sum_{j=a}^{B}W_{a,j}(0)+\sum_{k=0}^{a-1}Q_a^{[k]}(0)\right)(1-\tilde{S}_a(\theta)) \\
& +\left\{1-\sum_{m=n}^{a-1}e_{m,n}\left(\sum_{r=a}^{B}g_{r-m}\sum_{i=r}^{B}y_i\tilde{S}_r(\theta)+\sum_{l=1}^{\infty}\sum_{r=a}^{B}g_{r-m+l}y_r\tilde{S}_r(\theta)\right)\right\} \\
& (1-\delta)\sum_{n=0}^{a-1}\sum_{k=0}^{n}Q_n^{[k]}(0)+\sum_{n=0}^{\infty}\sum_{r=a}^{B}P_{n,r}(0)\alpha(1-\tilde{S}(\theta)).
\end{aligned}$$

Taking $\theta \to 0$ in (53) and using Lemma 2, (25), (27), (29), L'Hôspital's rule, and the normalization condition $(1-\delta)\sum_{n=0}^{a-1}R_n+\sum_{n=0}^{\infty}\sum_{r=a}^{B}(P_{n,r}+W_{n,r})+\sum_{n=0}^{\infty}\sum_{k=0}^{min(n,a-1)}Q_n^{[k]}$=1, after few simplification we get desired outcome.

Lemma 4.

$$W^{+}(z) \;=\; P^{+}(z)M_{os}(z)\alpha. \tag{54}$$

Proof. Multiplying (22)-(23) by proper power of z and y and summing over the range of n and r we obtain

$$\begin{aligned}
(\lambda-\theta-\lambda X(z))W(z,y,\theta) \;=\; & \sum_{n=0}^{\infty}\sum_{r=a}^{B}P_{n,r}(0)\alpha\tilde{S}(\theta)z^n y^r \\
& -\sum_{n=0}^{\infty}\sum_{r=a}^{B}W_{n,r}(0)z^n y^r.
\end{aligned} \tag{55}$$

Substituting $\theta = \lambda - \lambda X(z)$ in the above expression and using Lemma 2 and (43) we get

$$W^{+}(z,y) \ = \ \sum_{n=0}^{\infty}\sum_{r=a}^{B} P_{n,r}^{+}\alpha M_{os}(z) z^n y^r. \tag{56}$$

Setting $y = 1$ in (56) and using (25) and (32) we get desired result (54).

Lemma 5.

$$W_n^{+} \ = \ \alpha \sum_{i=0}^{n} P_i^{+} q_{n-i}, \ n \geq 0, a \leq r \leq B. \tag{57}$$

Proof. Using (34), (32), and (43) in (54), after simplification we get

$$\sum_{n=0}^{\infty} W_n^{+} z^n = \alpha \sum_{n=0}^{\infty}\sum_{i=0}^{n} P_i^{+} q_{n-i} z^n. \tag{58}$$

Now collecting the coefficients of z^n $(n \geq 0)$ from both the side of (58) we get desired outcome (57).

Lemma 6.

$$Q^{+}(z) = \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q_n^{[k]+} z^n = \sum_{k=0}^{a-1} (P_k^{+}(1-\alpha) + W_k^{+} + \delta Q_k^{+}) N^{(k)}(z) z^k. \tag{59}$$

Proof. Multiplying (20) and (21) by proper power of z and y and summing them over the range of n and k, we get

$$\begin{aligned}(\lambda - \theta - \lambda X(z)) Q(z,y,\theta) \ &= \ \sum_{k=0}^{a-1} \left(\sum_{r=a}^{B} (P_{k,r}(0)(1-\alpha) + W_{k,r}(0)) + \delta \sum_{j=0}^{k} Q_j^{[k]}(0) \right) \\ &\quad \tilde{V}_k(\theta) z^k y^k - \sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q_n^{[k]}(0) z^n y^k. \end{aligned} \tag{60}$$

Now substituting $\theta = \lambda - \lambda X(z)$ in (60) and using Lemma 2, (25), (27), (29), and (44) we obtain

$$\sum_{k=0}^{a-1}\sum_{n=k}^{\infty} Q_n^{[k]+} z^n y^k = \sum_{k=0}^{a-1} \left(P_k^{+}(1-\alpha) + W_k^{+} + \delta Q_k^{+}\right) N^{(k)}(z) z^k y^k. \tag{61}$$

Substituting $y = 1$ in (61) we obtain desired result.

Lemma 7.

$$Q_n^{[k]+} = \left(P_k^{+}(1-\alpha) + W_k^{+} + \delta \sum_{j=0}^{k} Q_k^{[j]+} \right) w_{n-k}^{(k)}, \ \ 0 \leq k \leq a-1, n \geq k. \tag{62}$$

Proof. From (61) collecting the coefficients of y^k $(0 \le k \le a-1)$ we obtain,

$$\sum_{n=k}^{\infty} Q_n^{[k]+} z^n = (P_k^+(1-\alpha) + W_k^+ + \delta Q_k^+) N^{(k)}(z) z^k. \tag{63}$$

Now using (44) and (29) in (63) and collecting the coefficients of z^n $(n \ge k)$ we obtain desired result (62).

Hence from Lemma 7 it is clear that once P_k^+ $(0 \le k \le a-1)$ are known, the joint probabilities $Q_n^{[k]+}$ $(0 \le k \le a-1, n \ge k)$ are also known.

Multiplying (18)-(19) by proper power of z and y and summing over the range of n and r we obtain

$$\begin{aligned}
&(\lambda - \theta - \lambda X(z)) P(z, y, \theta) = \\
&\quad \sum_{r=a}^{B} \left(\sum_{k=0}^{a-1} Q_r^{[k]}(0) + \sum_{j=a}^{B} (P_{r,j}(0)(1-\alpha) + W_{r,j}(0)) \right) \sum_{i=r}^{B} y_i \tilde{S}_r(\theta) y^r \\
&\quad + \sum_{n=1}^{\infty} \sum_{r=a}^{B} \left(\sum_{k=0}^{a-1} Q_{n+r}^{[k]}(0) + \sum_{j=a}^{B} P_{n+r,j}(0)(1-\alpha) \right. \\
&\quad \left. + \sum_{j=a}^{B} W_{n+r,j}(0) \right) \tilde{S}_r(\theta) y_r z^n y^r + (1-\delta)\lambda \sum_{r=a}^{B} \sum_{j=0}^{a-1} R_j g_{r-j} \sum_{i=r}^{B} y_i \tilde{S}_r(\theta) y^r \\
&\quad + (1-\delta)\lambda \sum_{n=1}^{\infty} \sum_{r=a}^{B} \sum_{j=0}^{a-1} R_j g_{n+r-j} y_r \tilde{S}_r(\theta) z^n y^r - \sum_{n=0}^{\infty} \sum_{r=a}^{B} P_{n,r}(0) z^n y^r.
\end{aligned}$$

Substituting $\theta = \lambda - \lambda X(z)$ in the above expression and using Lemma 1, Lemma 2, (25), (27), (29), (31), and (42) we get

$$\begin{aligned}
P^+(z, y) &= \sum_{r=a}^{B} \beta_r^+ \sum_{i=r}^{B} y_i M^{(r)}(z) y^r + \sum_{n=1}^{\infty} \sum_{r=a}^{B} \beta_{n+r}^+ M^{(r)}(z) y_r z^n y^r \\
&\quad + (1-\delta) \sum_{r=a}^{B} \sum_{j=0}^{a-1} \sum_{m=0}^{j} Q_m^+ e_{j,m} \left(g_{r-j} \sum_{i=r}^{B} y_i + \sum_{n=1}^{\infty} g_{n+r-j} y_r z^n \right) M^{(r)}(z) y^r. \\
&\quad .
\end{aligned} \tag{64}$$

Substituting $y = 1$ in (64) and using Lemma 6, (54) and (32) after some algebraic manipulation we get following result

$$P^+(z) = \frac{\vec{E}}{K(z)}, \tag{65}$$

where

$$\begin{aligned}\vec{E} &= z^B\sum_{r=a}^{B-1}\beta_r^+\sum_{i=r}^{B}y_iM^{(r)}(z)-\sum_{i=a}^{B-1}y_iM^{(i)}(z)\sum_{n=0}^{i}\beta_n^+z^{n+B-i}-\sum_{n=0}^{B-1}\beta_n^+z^ny_BM^{(B)}(z)\\&+\sum_{i=a}^{B}y_iM^{(i)}(z)\sum_{k=0}^{a-1}(P_k^+(1-\alpha)+W_k^++\delta Q_k^+)N^{(k)}(z)z^{B-i+k}\\&+(1-\delta)\sum_{n=0}^{a-1}Q_n^+\sum_{m=n}^{a-1}e_{m,n}\bigg(z^B\sum_{r=a}^{B}g_{r-m}\sum_{i=r}^{B}y_iM^{(r)}(z)\\&+\sum_{i=a}^{B}y_iM^{(i)}(z)z^{B-i+m}(X(z)-\sum_{n=1}^{i-m}g_nz^n)\bigg),\end{aligned}$$

and $K(z)=z^B-(1-\alpha+\alpha M_{os}(z))\sum_{i=a}^{B}y_iM^{(i)}(z)z^{B-i}$.

Finally, using (65) in (64) after some algebraic manipulation we obtain

$$P^+(z,y)=\frac{\bigwedge(z,y)}{K(z)}, \tag{66}$$

where

$$\begin{aligned}\bigwedge(z,y) &= \sum_{r=a}^{B-1}\beta_r^+\sum_{i=r}^{B}y_iM^{(r)}(z)\bigg(y^rK(z)+(1-\alpha+\alpha M_{os}(z))\sum_{j=a}^{B}y_jM^{(j)}(z)z^{B-j}y^j\bigg)\\&+\sum_{i=a}^{B-1}y_iM^{(i)}(z)z^{-i}\sum_{n=0}^{i}\beta_n^+z^n\bigg(-y^iK(z)-(1-\alpha+\alpha M_{os}(z))z^B\\&\sum_{j=a}^{B}y_jM^{(j)}(z)z^{-j}y^j\bigg)+\sum_{n=0}^{B-1}\beta_n^+z^ny_BM^{(B)}(z)\bigg(-z^{-B}y^BK(z)\\&-\sum_{j=a}^{B}y_jM^{(j)}(z)z^{-j}y^j\bigg)+\sum_{k=0}^{a-1}(P_k^+(1-\alpha)+W_k^++\delta Q_k^+)N^{(k)}(z)z^k\\&\sum_{j=a}^{B}y_jM^{(j)}(z)z^{B-j}y^j+(1-\delta)\sum_{n=0}^{a-1}Q_n^+\sum_{m=n}^{a-1}e_{m,n}\bigg\{\bigg(g_{r-m}\sum_{i=r}^{B}y_iM^{(r)}(z)y^r\\&+\sum_{r=a}^{B}y_rM^{(r)}(z)z^{-r+m}y^r(X(z)-\sum_{n=1}^{i-m}g_nz^n)\bigg)K(z)+(1-\alpha+\alpha M_{os}(z))\\&\sum_{r=a}^{B}y_rM^{(r)}(z)z^{B-r}y^r\big(\sum_{l=a}^{B}g_{l-m}\sum_{i=l}^{B}y_iM^{(l)}(z)\end{aligned}$$

$$+\sum_{i=a}^{B} y_i M^{(i)}(z) z^{-(i-m)}(X(z) - \sum_{n=1}^{i-m} g_n z^n)\Big)\Big\}.$$

It may be observed from (66) that the generating function $P^+(z,y)$ has been expressed in compact form, except for the B unknowns $\{P_n^+\}_{n=0}^{B-1}$. One can further note that from Lemma 7 once P_k^+ $(0 \le k \le a-1)$ are known then the joint probabilities $Q_n^{[k]+}$ $(0 \le k \le a-1)$ are completely known. Hence, to find $P_{n,r}^+$ $(a \le r \le B, n \ge 0)$ and $Q_n^{[k]+}$ $(0 \le k \le a-1, n \ge k)$ we should find the unknowns $\{P_n^+\}_{n=0}^{B-1}$. Next section is dedicated in getting these unknowns $\{P_n^+\}_{n=0}^{B-1}$.

3.2. Procedure of getting the unknowns P_n^+ $(0 \le n \le B-1)$

It can be seen that the unknowns P_n^+ $(0 \le n \le B-1)$ as appeared in (66) are same as the unknowns which are appeared in (65). Using the result given in Abolnikov and Dukhovny ([1], Theorem 4.1 and Lemma 4.1, page 341) for $\frac{\lambda\tilde{g}\sum_{i=a}^{B}\frac{y_i}{\mu_i}+\lambda\tilde{g}\frac{\alpha}{\mu}}{\tilde{y}} < 1$, $K(z)$ has $(B-1)$ zeros say $x_1, x_2, ..., x_l$ with multiplicity $r_1, r_2, ..., r_l$, respectively, inside the unit circle $|z| = 1$ $\big($where $(l \le B-1)$ and $\sum_{i=1}^{l} r_i = (B-1)\big)$ and one simple zero, say, $z_B = 1$, on the boundary of unit circle $|z|$=1. Due to analyticity of (65) in $|z| \le 1$ these zeros are also the zeros of numerator of (65). Hence, from (65) we have $(B-1)$ linearly independent equations,

$$\Bigg[\frac{d^{i-1}}{dz^{i-1}}\Big\{z^B\sum_{r=a}^{B-1}\beta_r^+\sum_{i=r}^{B} y_i M^{(r)}(z) - \sum_{i=a}^{B-1} y_i M^{(i)}(z)\sum_{n=0}^{i}\beta_n^+ z^{n+B-i} - \sum_{n=0}^{B-1}\beta_n^+ z^n y_B M^{(B)}(z)$$
$$+\sum_{i=a}^{B} y_i M^{(i)}(z)\sum_{k=0}^{a-1}(P_k^+(1-\alpha) + W_k^+ + \delta Q_k^+)N^{(k)}(z) z^{B-i+k} + (1-\delta)\sum_{n=0}^{a-1} Q_n^+ \sum_{m=n}^{a-1} e_{m,n}$$
$$\Big(z^B\sum_{r=a}^{B} g_{r-m}\sum_{i=r}^{B} y_i M^{(r)}(z) + \sum_{i=a}^{B} y_i M^{(i)}(z) z^{B-i+m}(X(z) - \sum_{n=1}^{i-m} g_n z^n)\Big)\Big\}\Bigg]_{z=x_j} = 0,$$
$$1 \le j \le l \;\;\&\;\; 1 \le i \le r_j, \quad (67)$$

where $\frac{d^0}{dz^0}h(z) = h(z)$.
Now using (66), Lemma 6 and the normalization condition $(1+\alpha)P^+(1) + Q^+(1) = 1$, after applying L'Hôspital's rule, we get

$$(1+\alpha)\Big(\sum_{r=a}^{B-1}\beta_r^+\sum_{i=r}^{B} y_i(B + \lambda\tilde{g}s_r) - \sum_{i=a}^{B} y_i \sum_{n=0}^{i}\beta_n^+(n + B - i + \lambda\tilde{g}s_i)$$
$$-\sum_{n=0}^{B-1}\beta_n^+ y_B(n + \lambda\tilde{g}s_B)) + (1-\delta)\sum_{n=0}^{a-1} Q_n^+ \sum_{m=n}^{a-1} e_{m,n}\big(\sum_{r=a}^{B} g_{r-m}\sum_{i=r}^{B} y_i(\lambda\tilde{g}s_r + B)$$
$$+\sum_{i=a}^{B} y_i(\tilde{g} - \sum_{n=1}^{i-m} n g_n) + (1 - \sum_{n=1}^{i-m} n g_n)(B - i + m + \lambda\tilde{g}s_i))\Big)$$

$$+\sum_{i=a}^{B}\sum_{k=0}^{a-1}(P_k^+(1-\alpha)+W_k^++\delta Q_k^+)(\lambda\tilde{g}(s_i+x_k)+B-i+k)=\tilde{y}(1-\rho)$$

Hence, (67) and (3.2) together forms non-homogenous system of B linearly independent equations in B unknowns P_n^+ $(0\leq n\leq B-1)$, solving them we uniquely determine P_n^+ $(0\leq n\leq B-1)$.

Now using (31) in (66) and then collecting the coefficients of y^r $(a\leq r\leq B)$ we get

$$\sum_{n=0}^{\infty}P_{n,r}^+z^n=\frac{\vec{O}(z,r)}{K(z)},a\leq r\leq B-1, \tag{68}$$

where

$$\begin{aligned}\vec{O}(z,r) &= \beta_r^+\sum_{i=r}^{B}y_iM^{(r)}(z)K(z)+(1-\alpha+\alpha M_{os}(z))y_rM^{(r)}(z)z^{B-r}\\ &\sum_{j=a}^{B-1}\beta_j^+\sum_{i=j}^{B}y_iM^{(i)}(z)-y_rM^{(r)}(z)z^{-r}\sum_{n=0}^{r}\beta_n^+z^nK(z)\\ &-(1-\alpha+\alpha M_{os}(z))\sum_{i=a}^{B-1}y_iM^{(i)}(z)\sum_{n=0}^{i}\beta_n^+z^{n+B-r-i}\\ &y_rM^{(r)}(z)-(1-\alpha+\alpha M_{os}(z))\sum_{n=0}^{B-1}\beta_n^+z^ny_BM^{(B)}(z)y_rM^{(r)}(z)z^{-r}\\ &+\sum_{k=0}^{a-1}(P_k^+(1-\alpha)+W_k^++\delta Q_k^+)N^{(k)}(z)z^ky_rM^{(r)}(z)z^{B-r}\\ &+(1-\delta)\sum_{n=0}^{a-1}Q_n^+\sum_{m=n}^{a-1}e_{m,n}\Bigg\{\Bigg(g_{r-m}\sum_{i=r}^{B}y_iM^{(r)}(z)+y_rM^{(r)}(z)z^{-r+m}(X(z)\\ &-\sum_{n=1}^{r-m}g_nz^n)\Bigg)K(z)+(1-\alpha+\alpha M_{os}(z))y_rM^{(r)}(z)z^{B-r}\\ &\Bigg(\sum_{l=a}^{B}g_{l-m}\sum_{i=l}^{B}y_iM^{(l)}(z)+\sum_{i=a}^{B}y_iM^{(i)}(z)z^{-(i-m)}(X(z)-\sum_{n=1}^{i-m}g_nz^n)\Bigg)\Bigg\}.\end{aligned}$$

$$\sum_{n=0}^{\infty}P_{n,B}^+z^n=\frac{\vec{O}(z,b)}{K(z)}, \tag{69}$$

where

$$\vec{O}(z,b) = (1-\alpha+\alpha M_{os}(z))y_BM^{(B)}(z)\sum_{j=a}^{B-1}\beta_j^+\sum_{i=j}^{B}y_iM^{(i)}(z)-(1-\alpha+\alpha M_{os}(z))$$

$$\sum_{i=a}^{B-1} y_i M^{(i)}(z) \sum_{n=0}^{i} \beta_n^+ z^{n-i} y_B M^{(B)}(z) - \sum_{n=0}^{B-1} \beta_n^+ z^n y_B M^B(z) z^{-B} \big(K(z)$$
$$+(1-\alpha+\alpha M_{os}(z)) y_B M^{(B)}(z)\big) + \sum_{k=0}^{a-1} (P_k^+(1-\alpha) + W_k^+ + \delta Q_k^+)$$
$$N^{(k)}(z) z^k y_B M^{(B)}(z) + (1-\delta) \sum_{n=0}^{a-1} Q_n^+ \sum_{m=n}^{a-1} e_{m,n} \Bigg(g_{B-m} y_B M^{(B)}(z)$$
$$+y_B M^{(B)}(z) z^{-B+m} (X(z) - \sum_{n=1}^{B-m} g_n z^n) \Bigg) K(z)$$
$$+(1-\delta) \sum_{n=0}^{a-1} Q_n^+ \sum_{m=n}^{a-1} e_{m,n} (1-\alpha+\alpha M_{os}(z)) y_B M^{(B)}(z)$$
$$\Bigg(\sum_{l=a}^{B} g_{l-m} \sum_{i=l}^{B} y_i M^{(l)}(z) + \sum_{i=a}^{B} y_i M^{(i)}(z) z^{-(i-m)} (X(z) - \sum_{n=1}^{i-m} g_n z^n) \Bigg).$$

For further investigation, we assume that the LST of the vacation time distribution and service time distribution is a rational function, i.e., $\tilde{V}_k(\theta) = \frac{F_k(\theta)}{C_k(\theta)}$, $\tilde{S}_r(\theta) = \frac{F_r(\theta)}{C_r(\theta)}$, and $\tilde{S}(\theta) = \frac{\hat{F}(\theta)}{\hat{C}(\theta)}$ where $F_k(\theta), C_k(\theta), F_r(\theta)$, $C_r(\theta)$, $\hat{F}(\theta)$, and $\hat{C}(\theta)$ are polynomials in θ. Even distribution functions with transcendental LST can be rationalized by pad$\acute{e}$ approximation. Substituting $\tilde{V}_k(\lambda - \lambda X(z)) = \frac{F_k(\lambda-\lambda X(z))}{C_k(\lambda-\lambda X(z))}$, $0 \le k \le a-1$, $\tilde{S}_r(\lambda - \lambda X(z)) = \frac{F_r(\lambda-\lambda X(z))}{C_r(\lambda-\lambda X(z))}$, $a \le r \le B$, and $S(\lambda - \lambda X(z)) = \frac{\hat{F}(\lambda-\lambda X(z))}{\hat{C}(\lambda-\lambda X(z))}$ in (68)-(69) after some simplification we obtain

$$\sum_{n=0}^{\infty} P_{n,r}^+ z^n = \frac{L_r(z)}{D_r(z)},\ a \le r \le B. \tag{70}$$

Where $L_r(z)$ and $D_r(z)$ are polynomials of degree u_r and d_r, respectively, and $D_r(z)$ is monic. We will now apply the partial fraction method to (70) for obtaining the joint probabilities $P_{n,r}$ $(n \ge 0, a \le r \le B)$. Since (70) is analytic in $|z| \le 1$, therefore, the zeros of $D_r(z)$ $(a \le r \le B)$ lying inside and on the unit circle $|z| = 1$ do not play any role in getting $P_{n,r}$ $(n \ge 0, a \le r \le B)$. Hence, in order to obtain all the joint probabilities $P_{n,r}$ $(n \ge 0, a \le r \le B)$ we need to know about all the zeros of $D_r(z)$ $(a \le r \le B)$ of modulus greater than one. Let $\gamma_{1,r}, \gamma_{2,r}, ..., \gamma_{l_r,r}$ be the zeros of $D_r(z)$ of modules greater than one with multiplicity $\tau_{1,r}, \tau_{2,r}, ..., \tau_{l_r,r}$, respectively, such that $\sum_{j=1}^{l_r} \tau_{j,r} \le d_r$. We discuss here the following two cases:

Case I : $d_r \le u_r$

Applying the partial fraction method to the right hand side of (70) we obtain

$$\sum_{n=0}^{\infty} P^{+}_{n,r} z^n = \sum_{i=0}^{u_r-d_r} \varrho_i z^i + \sum_{j=1}^{l_r}\sum_{i=1}^{\tau_{j,r}} \frac{B_{i,j,r}}{(z-\gamma_{j,r})^{\tau_{j,r}-i+1}}, \tag{71}$$

where

$$B_{i,j,r} = \frac{1}{(i-1)!}\left[\frac{d^{i-1}}{dz^{i-1}}\left(\frac{L_r(z)\frac{d^{\tau_{j,r}}}{dz^{\tau_{j,r}}}(z-\gamma_{j,r})^{\tau_{j,r}}}{\frac{d^{\tau_{j,r}}}{dz^{\tau_{j,r}}}(D_r(z))}\right)\right]_{z=\gamma_{j,r}}, \tag{72}$$
$$a \le r \le B, j = 1,2,...,l_r,\ i = 1,2,...,\tau_{j,r}.$$

Accumulating the coefficients of z^n $(n \ge 0)$ from both side of (71), we obtain for $a \le r \le B$,

$$P^{+}_{n,r} = \begin{cases} \left(\varrho_n + \sum\limits_{j=1}^{l_r}\sum\limits_{i=1}^{\tau_{j,r}} \frac{B_{i,j,r}}{(-1)^{\tau_{j,r}-i+1}\gamma_{j,r}^{\tau_{j,r}+n-i+1}}\binom{\tau_{j,r}-i+n}{\tau_{j,r}-i}\right), & 0 \le n \le u_r - d_r, \\ \left(\sum\limits_{j=1}^{l_r}\sum\limits_{i=1}^{\tau_{j,r}} \frac{B_{i,j,r}}{(-1)^{\tau_{j,r}-i+1}\gamma_{j,r}^{\tau_{j,r}+n-i+1}}\binom{\tau_{j,r}-i+n}{\tau_{j,r}-i}\right), & n > u_r - d_r. \end{cases} \tag{73}$$

Case II : $d_r > u_r$

We remove the first summation term of the right hand side of (71) and hence, for $a \le r \le B$, we obtain

$$P^{+}_{n,r} = \left(\sum_{j=1}^{l_r}\sum_{i=1}^{\tau_{j,r}} \frac{B_{i,j,r}}{(-1)^{\tau_{j,r}-i+1}\gamma_{j,r}^{\tau_{j,r}+n-i+1}}\binom{\tau_{j,r}-i+n}{\tau_{j,r}-i}\right), \ \ n \ge 0. \tag{74}$$

Equation (73) [or (74)] gives the joint probabilities of queue and server content at service (FES) completion epoch.

Theorem 1.

$$W^{+}_{n,r} \ = \ \alpha \sum_{i=0}^{n} P^{+}_{i,r} q_{n-i},\ n \ge 0, a \le r \le B. \tag{75}$$

Proof. Using (34), and collecting the coefficients of y^r $(a \le r \le B)$ from both the side of (56) we get

$$\sum_{n=0}^{\infty} W^{+}_{n,r} z^n \ = \ \sum_{n=0}^{\infty} P^{+}_{n,r} \alpha M_{os}(z) z^n \tag{76}$$

Using (43) in the above expression and collecting the coefficients of z^n $(n \ge 0)$ we get desired outcome.

Thus we complete here the evaluation of the joint probabilities of queue and server content at service (FES and SOS) completion epoch and the joint probabilities of queue length and vacation type at vacation termination epoch.

3.3. Joint probabilities at arbitrary epoch

In the previous section we have successfully achieved the joint probabilities of the queue and server content at service (FES and SOS) completion epoch, as well as the joint probabilities of the queue size and the vacation type at vacation termination epoch. In this section, we now turn our objective for getting these probabilities at arbitrary epoch.

Theorem 2. *The probabilities* R_n $(0 \leq n \leq a-1)$*,* $P_{n,r}$ *(*$W_{n,r}$*)* $(n \geq 0, a \leq r \leq B)$ *and* $Q_n^{[k]}$ $(n \geq k, 0 \leq k \leq a-1)$ *are given by,*

$$R_n = \frac{\sum\limits_{m=0}^{n} e_{n,m} Q_m^+}{E}, \;\; 0 \leq n \leq a-1 \;\; (exist \; only \; for \; SV), \tag{77}$$

$$P_{0,r} = \frac{\beta_r^+ (\sum\limits_{i=r}^{B} y_i) + (1-\delta) \sum\limits_{n=0}^{a-1} \sum\limits_{m=n}^{a-1} e_{m,n} g_{r-m} Q_n^+ \sum\limits_{i=r}^{B} y_i - P_{0,r}^+}{E}, \; n \geq 0, \tag{78}$$

$$P_{n,r} = \sum_{j=1}^{n} P_{n-j,r} g_j + \frac{\beta_{n+r}^+ y_r + (1-\delta) \sum\limits_{m=0}^{a-1} Q_m^+ \sum\limits_{j=m}^{a-1} e_{j,m} g_{n+r-j} y_r - P_{n,r}^+}{E}, \quad n \geq 1, a \leq r \leq B, \tag{79}$$

$$Q_k^{[k]} = \frac{P_k^+(1-\alpha) + W_k^+ + \delta Q_k^+ - Q_k^{[k]+}}{E}, \;\; 0 \leq k \leq a-1, \tag{80}$$

$$Q_n^{[k]} = \sum_{i=1}^{n-k} g_i Q_{n-i}^{[k]} - \frac{Q_n^{[k]+}}{E}, \;\; n \geq k+1, \; 0 \leq k \leq a-1, \tag{81}$$

$$W_{0,r} = \frac{P_{0,r}^+ \alpha - W_{0,r}^+}{E}, \; a \leq r \leq B, \tag{82}$$

$$W_{n,r} = \sum_{j=1}^{n} W_{n-j,r} g_j + \frac{P_{n,r}^+ \alpha - W_{n,r}^+}{E}, \; n \geq 1, \; a \leq r \leq B. \tag{83}$$

where $E = \lambda f + (1-\delta) \sum\limits_{n=0}^{a-1} \sum\limits_{m=0}^{n} e_{n,m} Q_m^+$,

Proof. Dividing (1) by σ^{-1} and using Lemma 2, Lemma 3 and (29), we obtain

$$R_0 = \frac{(1 - \sum\limits_{n=0}^{a-1} R_n) Q_0^+}{\lambda f}. \tag{84}$$

Similarly, from (45), we obtain

$$R_n = \frac{(1 - \sum\limits_{i=0}^{a-1} R_i) \sum\limits_{m=0}^{n} \sum\limits_{k=0}^{m} e_{n,m} Q_m^{[k]+}}{\lambda f}, \;\; 0 \leq n \leq a-1. \tag{85}$$

Using (84) in (85), we obtain

$$R_n = \frac{R_0}{Q_0^+}\sum_{m=0}^{n}\sum_{k=0}^{m} e_{n,m}Q_m^{[k]+}, \;\; 0 \le n \le a-1. \tag{86}$$

Using (86) in (84) after some algebraic manipulation, we obtain

$$R_0 = \frac{Q_0^+}{\lambda f + \sum\limits_{n=0}^{a-1}\sum\limits_{m=0}^{n}\sum\limits_{k=0}^{m} e_{n,m}Q_m^{[k]+}}. \tag{87}$$

Using (87) in (86), we obtain

$$R_n = \frac{\sum\limits_{m=0}^{n} e_{n,m}Q_m^+}{\lambda f + \sum\limits_{j=0}^{a-1}\sum\limits_{l=j}^{a-1} e_{l,j}Q_j^+}, \;\; 0 \le n \le a-1. \tag{88}$$

Setting $\theta = 0$ in (18)-(23), we get

$$\begin{aligned}
\lambda P_{0,r} &= \sum_{k=0}^{a-1} Q_r^{[k]}(0)\sum_{i=r}^{B} y_i + \sum_{j=a}^{B}(P_{r,j}(0)(1-\alpha)+W_{r,j}(0))\sum_{i=r}^{r} y_i \\
&\quad +(1-\delta)\lambda\sum_{j=0}^{a-1} R_j g_{r-j}\sum_{i=r}^{B} y_i - P_{0,r}(0), \;\; a+1 \le r \le B, &(89)\\
\lambda P_{n,r} &= \lambda\sum_{j=1}^{n} P_{n-j,r}g_j + \sum_{j=a}^{B}(P_{n+r,j}(0)(1-\alpha)+W_{n+r,j}(0))y_r + \sum_{k=0}^{a-1} Q_{n+r}^{[k]}(0)y_r \\
&\quad +\lambda\sum_{j=0}^{a-1} R_j g_{n+r-j}y_r - P_{n,r}(0), \;\; n \ge 1, \; a \le r \le B-1, &(90)\\
\lambda Q_k^{[k]} &= \sum_{r=a}^{B}(P_{k,r}(0)(1-\alpha)+W_{k,r}(0))+\delta\sum_{j=0}^{k} Q_k^{[j]}(0)-Q_k^{[k]}(0), 0 \le k \le a-1, &(91)\\
\lambda Q_n^{[k]} &= \lambda\sum_{i=1}^{n-k} g_i Q_{n-1}^{[k]} - Q_n^{[k]}(0), \;\; n \ge k+1, \; 0 \le k \le a-1, &(92)\\
\lambda W_{0,r} &= P_{0,r}(0)\alpha - W_{0,r}(0), \;\; a \le r \le B, &(93)\\
\lambda W_{n,r} &= \lambda\sum_{j=1}^{n} W_{n-j,r}g_j + P_{n,r}(0)\alpha - W_{n,r}(0), \;\; n \ge 1, a \le r \le B. &(94)
\end{aligned}$$

Dividing (89) by σ^{-1}, respectively, and then using Lemma 2, Lemma 3, (25) and (29), we obtain

$$P_{0,r} = \frac{\begin{array}{l}\big(1-(1-\delta)\sum\limits_{i=0}^{a-1} R_i\big)\big\{(P_r^{+}(1-\alpha)+W_r^{+}+Q_r^{+})\sum\limits_{i=r}^{B} y_i \\ \quad +(1-\delta)\sum\limits_{n=0}^{a-1}\sum\limits_{m=n}^{a-1} g_{r-m}e_{m,n}Q_n^{+}\sum\limits_{i=r}^{B} y_i - P_{0,r}^{+}\big\}\end{array}}{\lambda f}. \quad (95)$$

Using (85) in (95), we simply obtain (78).

Applying similar process for (90), (91), (92), (93), and (94) respectively, after some algebraic manipulation we get desired outcome (79), (80), (81), (82), and (83).

4. Performance measures

Performance measure is the procedure that collects the information of the system and helps the manager to run the system smoothly. Since all the steady state probabilities are known, the present section presents some important performance measures of the model under consideration.

1. Queue length (i.e., the number of customer in the queue without the number of customer with the server) distribution is given by,

$$P_n^{queue} = \begin{cases} (1-\delta)R_n + \sum\limits_{r=a}^{B}(P_{n,r}+W_{n,r}) + \sum\limits_{k=0}^{n} Q_n^{[k]},\ 0 \le n \le a-1, \\ \sum\limits_{r=a}^{B}(P_{n,r}+W_{n,r}) + \sum\limits_{k=0}^{a-1} Q_n^{[k]},\ n \ge a. \end{cases}$$

2. System length (i.e., the number of customer in the queue with the number of customer with the server) distribution is given by,

$$P_n^{system} = \begin{cases} R_n + \sum\limits_{k=0}^{n} Q_n^{[k]},\ 0 \le n \le a-1, \\ \sum\limits_{m=a}^{n}(P_{n-m,m}+W_{n-m,m}) + \sum\limits_{k=0}^{a-1} Q_n^{[k]},\ a \le n \le B, \\ \sum\limits_{r=a}^{B}(P_{n-r,r}+W_{n-r,r}) + \sum\limits_{k=0}^{a-1} Q_n^{[k]},\ n \ge B+1. \end{cases}$$

3. The server content distribution when the server is busy in FES, is given by,

$$FES_r^{ser} = \frac{\sum\limits_{n=0}^{\infty} P_{n,r}}{\sum\limits_{j=a}^{B}\sum\limits_{n=0}^{\infty} P_{n,j}},\ (a \le r \le B)\ .$$

4. The server content distribution when the server is busy in SOS, is given by,
$SOS_r^{ser}=\frac{\sum\limits_{n=0}^{\infty} W_{n,r}}{\sum\limits_{j=a}^{B}\sum\limits_{n=0}^{\infty} W_{n,j}}, (a \leq r \leq B)$.

5. The server content distribution when the server is busy, is given by,
$P_r^{ser}=\frac{\sum\limits_{n=0}^{\infty}(P_{n,r}+W_{n,r})}{\sum\limits_{j=a}^{B}\sum\limits_{n=0}^{\infty}(P_{n,j}+W_{n,j})}, (a \leq r \leq B)$.

6. Distribution of the vacation type when the server is on vacation is given by,
$Q_{vac}^{[k]}=\frac{\sum\limits_{n=k}^{\infty} Q_n^{[k]}}{\sum\limits_{l=0}^{a-1}\sum\limits_{n=l}^{\infty} Q_n^{[l]}}, 0 \leq k \leq a-1.$

7. The probability that the server is in a dormant state is $P_{dor}=(1-\delta)\sum\limits_{n=0}^{a-1} R_n$.

8. The probability that the server is busy is $P_{busy}=\sum\limits_{n=0}^{\infty}\sum\limits_{r=a}^{B}(P_{n,r}+W_{n,r})$.

9. The probability that the server is on vacation is $Q_{vac}=\sum\limits_{n=0}^{\infty}\sum\limits_{k=0}^{min(n,a-1)} Q_n^{[k]}$.

10. The probability that the server is idle is $P_{idle}=(1-\delta)P_{dor}+Q_{vac}$.

11. The expected number of customers in the queue (L_q) is given by
$L_q=(1-\delta)\sum\limits_{n=0}^{a-1} nR_n+\sum\limits_{n=0}^{\infty}\sum\limits_{r=a}^{B} n(P_{n,r}+W_{n,r})+\sum\limits_{k=0}^{a-1}\sum\limits_{n=k}^{\infty} nQ_n^{[k]}=(1-\delta)\sum\limits_{n=0}^{a-1} nP_n^{queue}+\sum\limits_{n=a-\delta a}^{\infty} nP_n^{queue}.$

12. The expected number of customers in the system (L_s) is given by
$L_s=(1-\delta)\sum\limits_{n=0}^{a-1} nR_n+\sum\limits_{n=0}^{\infty}\sum\limits_{r=a}^{B}(n+r)(P_{n,r}+W_{n,r})+\sum\limits_{k=0}^{a-1}\sum\limits_{n=k}^{\infty} nQ_n^{[k]}.$

13. The expected waiting time of a customer in the queue (W_q) is given by
$W_q=\frac{L_q}{\lambda\tilde{g}}$.

14. The expected waiting time of a customer in the system (W_s) is given by
$W_s=\frac{L_s}{\lambda\tilde{g}}$.

15. Expected number of customers with the server when server is busy (L^{ser}) is given by
$L^{ser}=\sum\limits_{r=a}^{B}(rP_r^{ser})$.

16. Expected vacation type taken by server when server is in vacation (L^{vac}) is given by $L^{vac} = \sum_{k=0}^{a-1} (kQ_{vac}^{[k]})$.

5. Numerical results

Through the use of several graphs and tables, we present a variety of numerical results in this section to illustrate the behavior of the performance measures of the model under investigation. First, we present some results in form of graphs (Figure 1 and Figure 2) based on the blood or swab testing procedure as presented in the introduction section. Figure 1 and Figure 2 represents the influence of the arrival rate of the samples on L_q and Q_{vac}, respectively, for SV and MV. Samples are coming in the health center in groups with probability $g_1 = 0.4$, $g_2 = 0.25$, $g_3 = 0.30$, $g_4 = 0.05$, $g_n = 0$, $n \geq 5$. Health worker tests a mixed sample during FES, and this mixed sample is formed by taking a group of samples from the queue. FES time follows Erlang (E_3) service time distribution with batch size dependent service rate $\mu_r = \frac{4.1}{r+1}$, $4 \leq r \leq 7$. If the mixed sample diagnosed positive then the health worker chooses mixed sample of 4 individual samples for the next test (FES), otherwise, he keeps the priority in mind to test mixed sample of 7 individual samples in the next test (FES). Probability that the test result of mixed sample is diagnosed positive is $\alpha = 0.3$. In case of positive diagnosed mixed sample in FES the health worker tests these mixed sample individually in a single kit treated as batch service (i.e., the health worker performs SOS) with E_3 distribution with a service rate of 1.5. Probability that the health worker test the mixed sample of the 4, 5, 6, and 7 individual samples is $y_4 = 0.3$, $y_5 = 0$, $y_6 = 0$, and $y_7 = 0.7$, respectively. Vacation time of the health worker follows Erlang (E_2) distribution with rate $\nu_k = \nu_{k-1} + 0.1$ $(1 \leq k \leq 3)$ where $\nu_0 = 0.2$. Since the increasing arrival rate increases the traffic, as a result, the expected queue length will also increase, which has been reflected in Figure 1 for both the vacation policy (SV and MV). Further, in MV policy chance that the server is on vacation is higher than for SV policy. As a result, in MV policy, the traffic in the system will be more than for SV policy. Thus, for a fixed value of λ in MV policy, the expected queue length is higher than in SV policy, which can also be seen in Figure 1. Thus, Figure 1 is on the expected direction. Further, if we increase the arrival rate, then the probability that the server is busy will also increase. Thus, the probability that the server is on vacation will decrease, which can be seen in Figure 2. As in MV policy, the server takes repeated vacations until it finds the required number of customers in the queue for service at the end of the vacation. Therefore for MV policy, the probability that the server is on vacation will be higher than for SV policy which can be seen in Figure 2.

For further justification of the presented model, we present Figure 3 and Figure 4 in which the the behavior of λ verses W_s and L^{vac} are reflected, respectively. The service time of each batch in FES follows a 2 - stage hyper-exponential (HE_2) distribution with a service rate of $\mu_r = (\frac{0.6}{\mu_{1,r}} + \frac{0.4}{\mu_{2,r}})^{-1}$ where $\mu_{j,r} = \frac{15}{3j+r}$, $1 \leq j \leq 2$, $3 \leq r \leq 6$. The service time in SOS follows an exponential distribution with a service rate of 5.5. The vacation time of the server follow the E_2 distribution with a vacation rate of $\nu_k = \nu_{k-1} + 2.1$, $1 \leq k \leq 2$ where

$\nu_0 = 4.2$. The other input parameters are taken as follows: $\alpha = 0.4$, $g_1 = 0.7$, $g_2 = 0.2$, $g_3 = 0.06$, $g_4 = 0.04$, $g_n = 0$, $n \geq 5$. $y_4 = 0.4$, $y_5 = 0$, $y_6 = 0$, $y_7 = 0.6$. Since increasing the arrival rate increases the traffic in the system, as a result, W_s and L^{vac} should also increase, which has been observed in Figure 3 and Figure 4. Further, in MV, server takes the repeated number of vacation therefore, in MV policy, traffic will be higher than for SV policy. Thus, for a fixed value of λ in MV policy, W_s and L^{vac} should be greater than for SV policy. Hence, Figure 3 and Figure 4 are on the expected direction.

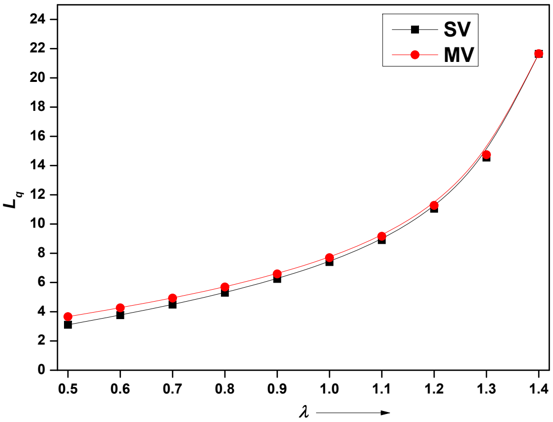

Figure 1. Effect of λ on L_q

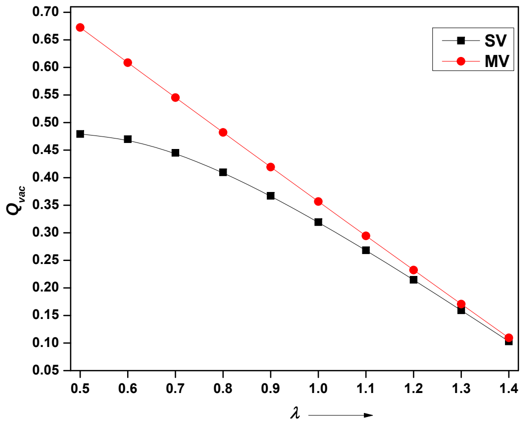

Figure 2. Effect of λ on Q_{vac}

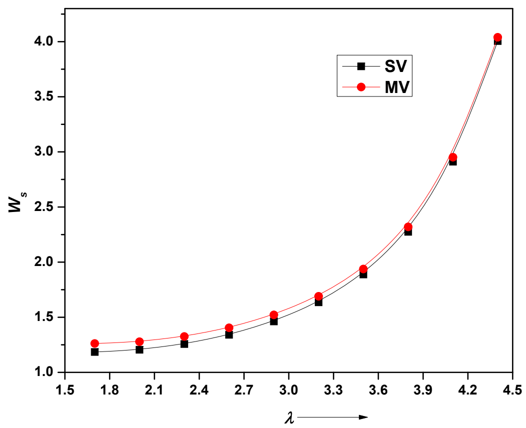

Figure 3. Effect of λ on W_s

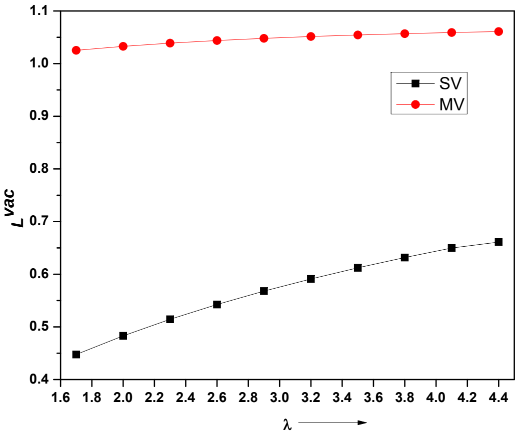

Figure 4. Effect of λ on L^{vac}

Table 1 and Table 2 present the steady state joint probabilities at service (FES and SOS) completion, vacation completion and arbitrary epoch for $M/G_r^{(a,Y)}/1$ queue with SOS and SV. Service time for both the FES and SOS follow the E_3 distribution, and vacation time follows E_2 distribution. The other input parameters are taken as $\lambda = 26.098$, $\mu_r = \frac{67.5}{r}$ $(4 \leq r \leq 7)$, $\mu = 17.5$, and $\nu_k = \nu_{k-1} + 2.35$ where $\nu_0 = 1.01$ $(1 \leq k \leq 3)$. $g_1 = 0.45$, $g_2 = 0.20$, $g_3 = 0.35$, $g_n = 0$ $(n \geq 4)$. $y_4 = 0.2$, $y_5 = 0$, $y_6 = 0$, $y_7 = 0.8$. The detail of Table 1 is given as follows:

- The first column presents the number of customers present in the queue (excluding the number in service).

- 2nd to 5th column present the joint probabilities of the queue and server content at service (FES) completion epoch.
- 6th to 9th column present the joint probabilities of the queue and server content at service (SOS) completion epoch.
- The 10th to 13th column presents the joint probabilities of the queue length and vacation type at the vacation termination epoch.
- 14th column presents the queue length distribution at service or vacation completion epoch.

The detail of Table 2 is given as follows:

- The first column presents the number of customers present in the queue (excluding the number in service).
- second column presents the probability that the system is in the state $(n, 0)$.
- 3rd to 6th column present the joint probabilities of queue and server content during FES at arbitrary epoch.
- 7th to 10th column present the joint probabilities of queue and server content during SOS at arbitrary epoch.
- 11th to 14th column present the joint probabilities of queue length and vacation type at arbitrary epoch.
- Last column presents the queue length distribution at arbitrary epoch.
- Performance measures (*viz.*, L_q, L_s, W_q, etc.) can be seen just below of the table.

Similarly, Table 3 presents the steady state joint probabilities at service (FES and SOS) completion epoch and vacation completion completion epoch for MV, and Table 4 presents the steady state joint probabilities at arbitrary epoch for $M/G_r^{(a,Y)}/1$ queue with SOS and MV. The input parameters, notations, and the service (vacation) time distribution are taken the same as taken for Table 1 and for Table 2.

Table 1. Steady state joint probabilities for $M^X/G_r^{(a,Y)}/1$ queue with SV and SOS at service (vacation) completion epoch

n	$P^+_{n,4}$	$P^+_{n,5}$	$P^+_{n,6}$	$P^+_{n,7}$	$W^+_{n,4}$	$W^+_{n,5}$	$W^+_{n,6}$	$W^+_{n,7}$	$Q_n^{[0]+}$	$Q_n^{[1]+}$	$Q_n^{[2]+}$	$Q_n^{[3]+}$	$P^+_n + Q^+_n$
0	0.0038	0.0024	0.0021	0.0016	0.0003	0.0002	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0109
1	0.0025	0.0013	0.0012	0.0027	0.0004	0.0002	0.0002	0.0003	0.0000	0.0003	0.0000	0.0000	0.0091
2	0.0025	0.0010	0.0010	0.0037	0.0004	0.0002	0.0002	0.0005	0.0000	0.0002	0.0007	0.0000	0.0105
3	0.0034	0.0015	0.0016	0.0051	0.0007	0.0003	0.0003	0.0008	0.0001	0.0002	0.0004	0.0015	0.0158
4	0.0030	0.0010	0.0011	0.0062	0.0007	0.0003	0.0003	0.0010	0.0001	0.0003	0.0004	0.0008	0.0152
5	0.0030	0.0008	0.0009	0.0071	0.0007	0.0003	0.0003	0.0013	0.0001	0.0003	0.0006	0.0007	0.0160
15	0.0029	0.0000	0.0001	0.0108	0.0009	0.0001	0.0001	0.0031	0.0001	0.0002	0.0002	0.0003	0.0186
16	0.0028	0.0000	0.0001	0.0107	0.0009	0.0000	0.0001	0.0031	0.0001	0.0002	0.0002	0.0002	0.0185
29	0.0022	0.0000	0.0000	0.0085	0.0007	0.0000	0.0000	0.0027	0.0001	0.0001	0.0000	0.0000	0.0143
30	0.0021	0.0000	0.0000	0.0083	0.0007	0.0000	0.0000	0.0027	0.0001	0.0001	0.0000	0.0000	0.0139
31	0.0021	0.0000	0.0000	0.0081	0.0007	0.0000	0.0000	0.0026	0.0001	0.0001	0.0000	0.0000	0.0136
51	0.0012	0.0000	0.0000	0.0047	0.0004	0.0000	0.0000	0.0015	0.0001	0.0000	0.0000	0.0000	0.0080
52	0.0012	0.0000	0.0000	0.0046	0.0004	0.0000	0.0000	0.0015	0.0001	0.0000	0.0000	0.0000	0.0077
75	0.0006	0.0000	0.0000	0.0024	0.0002	0.0000	0.0000	0.0008	0.0000	0.0000	0.0000	0.0000	0.0041
76	0.0006	0.0000	0.0000	0.0023	0.0002	0.0000	0.0000	0.0008	0.0000	0.0000	0.0000	0.0000	0.0039
77	0.0006	0.0000	0.0000	0.0023	0.0002	0.0000	0.0000	0.0007	0.0000	0.0000	0.0000	0.0000	0.0038
135	0.0001	0.0000	0.0000	0.0004	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0007
136	0.0001	0.0000	0.0000	0.0004	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0006
155	0.0001	0.0000	0.0000	0.0002	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0004
156	0.0001	0.0000	0.0000	0.0002	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0003
197	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
198	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
212	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
213	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
$\geq$ 214	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
sum	0.1596	0.0107	0.0118	0.5625	0.0479	0.0032	0.0035	0.1687	0.0079	0.0065	0.0071	0.0102	0.9996

Table 2. Steady state joint probabilities for $M^X/G_r^{(a,Y)}/1$ queue with SV and SOS at arbitrary epoch

n	R_n	$P_{n,4}$	$P_{n,5}$	$P_{n,6}$	$P_{n,7}$	$W_{n,4}$	$W_{n,5}$	$W_{n,6}$	$W_{n,7}$	$Q_n^{[0]}$	$Q_n^{[1]}$	$Q_n^{[2]}$	$Q_n^{[3]}$	P_n^{queue}
0	0.00002	0.0038	0.0034	0.0039	0.0038	0.0003	0.0002	0.0002	0.0001	0.0032				0.0190
1	0.00013	0.0018	0.0010	0.0013	0.0052	0.0003	0.0002	0.0002	0.0003	0.0014	0.0025			0.0142
2	0.00042	0.0017	0.0007	0.0009	0.0064	0.0003	0.0002	0.0001	0.0004	0.0013	0.0011	0.0026		0.0161
3	0.00109	0.0022	0.0011	0.0014	0.0080	0.0005	0.0002	0.0002	0.0006	0.0019	0.0009	0.0010	0.0035	0.0226
4		0.0019	0.0006	0.0008	0.0090	0.0005	0.0002	0.0002	0.0008	0.0016	0.0014	0.0008	0.0012	0.0189
5		0.0019	0.0004	0.0006	0.0098	0.0005	0.0002	0.0002	0.0009	0.0015	0.0011	0.0013	0.0010	0.0192
21		0.0016	0.0000	0.0000	0.0107	0.0005	0.0000	0.0000	0.0019	0.0013	0.0003	0.0001	0.0001	0.0164
22		0.0016	0.0000	0.0000	0.0105	0.0005	0.0000	0.0000	0.0018	0.0012	0.0003	0.0001	0.0001	0.0161
51		0.0007	0.0000	0.0000	0.0049	0.0002	0.0000	0.0000	0.0009	0.0006	0.0000	0.0000	0.0000	0.0074
52		0.0007	0.0000	0.0000	0.0048	0.0002	0.0000	0.0000	0.0009	0.0006	0.0000	0.0000	0.0000	0.0072
64		0.0005	0.0000	0.0000	0.0034	0.0002	0.0000	0.0000	0.0006	0.0004	0.0000	0.0000	0.0000	0.0052
65		0.0005	0.0000	0.0000	0.0033	0.0002	0.0000	0.0000	0.0006	0.0004	0.0000	0.0000	0.0000	0.0050
88		0.0003	0.0000	0.0000	0.0017	0.0001	0.0000	0.0000	0.0003	0.0002	0.0000	0.0000	0.0000	0.0026
89		0.0002	0.0000	0.0000	0.0017	0.0001	0.0000	0.0000	0.0003	0.0002	0.0000	0.0000	0.0000	0.0025
126		0.0001	0.0000	0.0000	0.0005	0.0000	0.0000	0.0000	0.0001	0.0001	0.0000	0.0000	0.0000	0.0008
127		0.0001	0.0000	0.0000	0.0005	0.0000	0.0000	0.0000	0.0001	0.0001	0.0000	0.0000	0.0000	0.0008
147		0.0000	0.0000	0.0000	0.0003	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0004
148		0.0000	0.0000	0.0000	0.0003	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0004
198		0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
199		0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
209		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
210		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
$\geq$ 211		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
sum	0.0017	0.0998	0.0084	0.0111	0.6154	0.0289	0.0019	0.0021	0.1017	0.0822	0.0204	0.0131	0.0133	1.0000

L_q=37.016, L_s=42.681, W_q=0.747, W_s= 0.861, L^{ser}=6.517, L^{vac}= 0.671

Q_{vac}=0.129, P_{idle}=0.131, P_{busy}=0.869

Table 3. Steady state joint probabilities for $M^X/G_r^{(a,Y)}/1$ queue with MV and SOS at service (vacation) completion epoch

n	$P_{n,4}^+$	$P_{n,5}^+$	$P_{n,6}^+$	$P_{n,7}^+$	$W_{n,4}^+$	$W_{n,5}^+$	$W_{n,6}^+$	$W_{n,7}^+$	$Q_n^{[0]+}$	$Q_n^{[1]+}$	$Q_n^{[2]+}$	$Q_n^{[3]+}$	$P_n^+ + Q_n^+$
0	0.0035	0.0023	0.0020	0.0017	0.0003	0.0002	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.0104
1	0.0023	0.0012	0.0012	0.0028	0.0003	0.0002	0.0002	0.0003	0.0000	0.0003	0.0000	0.0000	0.0089
2	0.0024	0.0010	0.0010	0.0038	0.0004	0.0002	0.0002	0.0005	0.0000	0.0002	0.0007	0.0000	0.0104
3	0.0032	0.0015	0.0015	0.0052	0.0006	0.0003	0.0003	0.0008	0.0001	0.0002	0.0005	0.0018	0.0160
4	0.0029	0.0010	0.0011	0.0064	0.0007	0.0003	0.0003	0.0011	0.0001	0.0003	0.0004	0.0010	0.0155
5	0.0029	0.0007	0.0009	0.0074	0.0007	0.0003	0.0003	0.0013	0.0001	0.0003	0.0006	0.0009	0.0164
15	0.0029	0.0000	0.0001	0.0111	0.0009	0.0000	0.0001	0.0032	0.0001	0.0002	0.0002	0.0003	0.0192
16	0.0029	0.0000	0.0001	0.0110	0.0009	0.0000	0.0001	0.0032	0.0001	0.0002	0.0002	0.0003	0.0190
29	0.0022	0.0000	0.0000	0.0085	0.0007	0.0000	0.0000	0.0027	0.0001	0.0001	0.0000	0.0000	0.0144
30	0.0021	0.0000	0.0000	0.0083	0.0007	0.0000	0.0000	0.0027	0.0001	0.0001	0.0000	0.0000	0.0140
31	0.0021	0.0000	0.0000	0.0081	0.0007	0.0000	0.0000	0.0026	0.0001	0.0001	0.0000	0.0000	0.0137
51	0.0012	0.0000	0.0000	0.0046	0.0004	0.0000	0.0000	0.0015	0.0001	0.0000	0.0000	0.0000	0.0078
52	0.0011	0.0000	0.0000	0.0045	0.0004	0.0000	0.0000	0.0015	0.0001	0.0000	0.0000	0.0000	0.0076
75	0.0006	0.0000	0.0000	0.0023	0.0002	0.0000	0.0000	0.0008	0.0000	0.0000	0.0000	0.0000	0.0039
76	0.0006	0.0000	0.0000	0.0023	0.0002	0.0000	0.0000	0.0007	0.0000	0.0000	0.0000	0.0000	0.0038
135	0.0001	0.0000	0.0000	0.0004	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0006
136	0.0001	0.0000	0.0000	0.0004	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0006
155	0.0001	0.0000	0.0000	0.0002	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0003
156	0.0000	0.0000	0.0000	0.0002	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0003
211	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
212	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
$\geq$ 213	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
sum	0.1581	0.0102	0.0114	0.5625	0.0474	0.0031	0.0034	0.1687	0.0075	0.0066	0.0080	0.0126	0.9996

Table 4. Steady state joint probabilities for $M^X/G_r^{(a,Y)}/1$ queue with MV and SOS at arbitrary epoch

n	$P_{n,4}$	$P_{n,5}$	$P_{n,6}$	$P_{n,7}$	$W_{n,4}$	$W_{n,5}$	$W_{n,6}$	$W_{n,7}$	$Q_n^{[0]}$	$Q_n^{[1]}$	$Q_n^{[2]}$	$Q_n^{[3]}$	P_n^{queue}
0	0.0035	0.0032	0.0038	0.0040	0.0003	0.0002	0.0002	0.0001	0.0030	0.0000	0.0000	0.0000	0.0183
1	0.0017	0.0010	0.0012	0.0054	0.0003	0.0002	0.0001	0.0003	0.0014	0.0026	0.0000	0.0000	0.0140
2	0.0016	0.0007	0.0009	0.0066	0.0003	0.0001	0.0001	0.0004	0.0012	0.0011	0.0029	0.0000	0.0160
3	0.0021	0.0010	0.0014	0.0083	0.0004	0.0002	0.0002	0.0006	0.0019	0.0009	0.0011	0.0044	0.0225
4	0.0019	0.0005	0.0008	0.0093	0.0004	0.0002	0.0002	0.0008	0.0015	0.0014	0.0009	0.0015	0.0195
5	0.0019	0.0004	0.0006	0.0101	0.0005	0.0002	0.0002	0.0010	0.0015	0.0011	0.0014	0.0012	0.0199
21	0.0016	0.0000	0.0000	0.0109	0.0005	0.0000	0.0000	0.0019	0.0012	0.0003	0.0001	0.0001	0.0167
22	0.0016	0.0000	0.0000	0.0107	0.0005	0.0000	0.0000	0.0019	0.0012	0.0003	0.0001	0.0001	0.0163
51	0.0007	0.0000	0.0000	0.0048	0.0002	0.0000	0.0000	0.0009	0.0006	0.0000	0.0000	0.0000	0.0073
52	0.0007	0.0000	0.0000	0.0047	0.0002	0.0000	0.0000	0.0009	0.0006	0.0000	0.0000	0.0000	0.0071
64	0.0005	0.0000	0.0000	0.0033	0.0002	0.0000	0.0000	0.0006	0.0004	0.0000	0.0000	0.0000	0.0050
65	0.0005	0.0000	0.0000	0.0032	0.0002	0.0000	0.0000	0.0006	0.0004	0.0000	0.0000	0.0000	0.0049
88	0.0002	0.0000	0.0000	0.0017	0.0001	0.0000	0.0000	0.0003	0.0002	0.0000	0.0000	0.0000	0.0025
89	0.0002	0.0000	0.0000	0.0016	0.0001	0.0000	0.0000	0.0003	0.0002	0.0000	0.0000	0.0000	0.0024
126	0.0001	0.0000	0.0000	0.0005	0.0000	0.0000	0.0000	0.0001	0.0001	0.0000	0.0000	0.0000	0.0008
127	0.0001	0.0000	0.0000	0.0005	0.0000	0.0000	0.0000	0.0001	0.0001	0.0000	0.0000	0.0000	0.0008
147	0.0000	0.0000	0.0000	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004
148	0.0000	0.0000	0.0000	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004
149	0.0000	0.0000	0.0000	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004
208	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
209	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
$\geq$ 210	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
sum	0.0991	0.0080	0.0107	0.6168	0.0287	0.0019	0.0021	0.1020	0.0788	0.0209	0.0147	0.0165	1.0000

L_q=36.349, L_s= 42.017, W_q= 0.733, W_s= .847, L^{ser}= 6.522, L^{vac}= 0.763
Q_{vac}= 0.131, P_{idle}=.0.131, P_{busy}=0.869

6. Conclusion

In this paper, we have analyzed the infinite-buffer bulk arrival batch size dependent bulk service queue and the queue length dependent SV (MV) with second optional service. The server operates the customer according to the (a, Y) rule. The fundamental mathematical analysis of the model includes mainly the supplementary variable techniques and bivariate generating function technique. The considered model analyzed the joint probabilities of the queue and server content at service completion (arbitrary) epoch and the joint probabilities of queue length and vacation type at vacation termination (arbitrary) epoch. Practical motivation and the numerical behavior of the considered model are also provided to validate our model in real life congestion control. The present model can be generalized with a more general arrival process (*viz.,* $BMAP$).

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