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A Note on Customer Joining Strategy in a Discrete-time *Geo* / *G* / 1 Queue with Server Vacations - a Simple Mean Value Analysis

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Abstract: This note considers customer choice behaviors in discrete-time queuing models with different information levels. In such models, customers try to maximize their utilities in deciding whether or not to enter the discrete-time Geo/G/1 queues with server's vacation times under two information cases. Using a simple mean value analysis, we construct the overall profit functions for customers and social welfare. Then, we analyze the customer equilibrium balking strategies and socially optimal balking strategies over different ranges of parameters. Numerical examples are presented to demonstrate the analytical results and to generate managerial insights for practitioners.

Keywords: Decision analysis, discrete-time, equilibrium strategy, information, vacation.

1. Introduction

Most classical queueing models assume that customers always join the queue upon their arrivals. However, if customers can make their decisions about joining or balking based on their own utilities (or benefits), then the queueing models should be analyzed under different information scenarios.

This study aims to investigate the performance of the discrete-time queueing system with customer choices of joining or balking from either the individual customer self-interest or social welfare perspective.

In the past two decades, there were many studies customer choice behaviors in queueing systems from an economic viewpoint, which has become a trend in the study of queueing systems. This research area was initiated by Naor [12] who studied the M/M/1 model with customer choice on joining or balking. Assuming an arriving customer would know the queue length (observable queue case), Naor [12] introduced the customer self-interest equilibrium and socially optimal strategies for a single server queue under a linear reward-cost structure. His study was complemented by Edelson and Hildebrand [5], who considered the

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same queueing system but assumed that the customers make their decisions without knowing the queue length (unobservable queue case). Since then, there was a growing number of papers that focused on the economic analysis of variants of the M/M/1 queues with the balking behaviors of customers, such as Hassin and Haviv [8] on M/M/1 queue with priorities, Burnetas and Economou [2] on M/M/1 queue with setup times, Guo and Zipkin [6] on M/M/1 queue with various levels of information and non-linear reward- cost structure, Hassin [7] on M/M/1 queue with various levels of information and uncertainty in the system parameters, Economou and Kanta [4] on M/M/1 queue with compartmented waiting space, Sun et al. [13] on queues with priority policies and Wang et.al [15–17] on the joining behavior of customers in variants of single server queues.

It is worth noting that most the previous studies are focused on the Markovian systems in which all random variables are exponentially distributed. Such a feature limits the application of the queueing models. Thus, the analysis on single server queues with generally distributed service times is practically relevant to queueing managers. However, the research on customers' joining strategy combined with general service times and server vacations is still limited. There are still queueing models worth exploration. Mandelbaum and Yechiali [11], Altman and Hassin [1], Haviv and Kerner [9] and Kerner [10] studied the customers' joining decisions in the classic M/G/1 models. Economou et al. [3] investigated the same class of models with server vacations. Zhang and Wang [18] discussed the pricing mechanism in an M/G/1 retrial queue. While these studies focused on continuous-time queueing models, we try to analyze the discrete-time queueing models with customer choice behaviors.

In this paper, we study the customers' joining/balking behaviors in the discrete-time queue with generally distributed service and vacation times. Two information scenarios are considered. These are the No-information (NI) case and Part-information (PI) case. For the NI case, an arriving customer does not know any information about the queueing system. For the PI case, an arriving customer knows the server state, but not the queue length.

The paper is organized as follows. Section 2 provides a mean value analysis for the Geo/G/1 queue with multiple vacations under the NI scenario and derives the customer self-interest equilibrium strategies as well as socially optimal strategies. Section 3 is devoted to the Part-information(PI) case with similar analysis as that in the NI case. In Section 4, we present numerical illustrations. Finally, Section 5 concludes the paper with a summary.

2. The NI case

2.1. Model formulation

Consider a discrete-time queueing system where the time axis is segmented into a sequence of equal time intervals (called slots). It is assumed that all events (arrivals and departures) occur at the slot boundaries, and therefore they may occur at the same time. For mathematical clarity, we define the arrival first (AF) system and assume that the departures occur at the instant immediately before the slot boundaries which is denoted by $t = n^-; n = 1, 2, \cdots$, and the arrivals occur at the moment immediately after the slot boundaries, i.e., $t = n^+; n = 0, 1, \cdots$. Customers arrive at the system according to a geometric arrival process with rate $p(0 < p < 1)$, that is, p is the probability that an arrival occurs in a slot. The inter-arrival interval time Y follows a geometric distribution with rate p.

$$P\{Y = j\} = p\overline{p}^{j-1}, \quad j = 1, 2, \cdots, \quad 0 < p < 1.$$

where $\overline{p} = 1 - p$. Then, the number of customers arriving during $[0, n]$, denoted by A_n, follows a negative binomial distribution:

$$P\{A_n = j\} = C_n^j p^j \overline{p}^{n-j}, \quad j = 0, 1, 2, \cdots, n.$$

The service time, denoted by B, follows a general distribution, which has the finite first and second moments, i.e., $E(B) < \infty; E(B^2) < \infty$. Its distribution function, PGF(probability generating function), and mean are represented, respectively, as:

$$b_j = P(B = j), j \geq 1, \quad G(z) = E(z^B) = \sum_{j=1}^{\infty} b_j z^j, \quad b = E[B] = G'(z)|_{z=1} = \sum_{j=1}^{\infty} j b_j.$$

Denote by R_B the residual service time, which has the probability distribution and mean:

$$q_k = \frac{1}{E[B]} \sum_{j=k+1}^{\infty} b_j, k = 0, 1, 2, \cdots, \qquad E[R_B] = \frac{E[B(B-1)]}{2E[B]}.$$

Such a system is denoted by Geo/G/1 queue. We focus on the Geo/G/1/MV where the server follows the multiple vacation(MV) policy. Under the multiple vacation poliy, the server starts a vacation whenever the system becomes empty and keeps taking the vacation until a vacation completion instant at which waiting customers exist in the system, then resumes serving customers. The vacation time, denoted by V, follows the general distribution function with the finite first and second moments, i.e., $E(V) < \infty; E(V^2) < \infty$. Its distribution function, PGF, and mean are represented respectively

$$v_j = P(V = j), j \geq 1, \quad V(z) = E(z^V) = \sum_{j=1}^{\infty} v_j z^j, \quad v = E[V] = V'(z)|_{z=1} = \sum_{j=1}^{\infty} j v_j.$$

Denote by R_V the residual vacation time, which has the probability distribution and mean:

$$w_k = \frac{1}{E[V]} \sum_{j=k+1}^{\infty} v_j, k = 0, 1, 2, \cdots, \qquad E[R_V] = \frac{E[V(V-1)]}{2E[V]}.$$

To model the customer's joining strategy, we assume that customer's utility equals a reward for receiving service, denoted by K, minus an expected waiting cost. Here, the waiting cost is a linear function of the waiting time. Let C be the cost per time unit and S be the sojourn time for the customer (i.e. waiting time in queue plus service time). Thus the expected waiting cost is $CE[S]$. To ensure that an arrival customer must enter an empty system, we assume

$$K > C(E[R_V] + E[B]), \tag{1}$$

which means that the service reward must be larger than the expected cost of joining an empty system with the server on vacation. To ensure the stability of the queueing system, we futher assume

$$pqE[B] < 1. \tag{2}$$

2.2. Equilibrium analysis under the NI scenario

In the NI case, an arriving customer has no information about the system (e.g. no information about the queue length and the server status). In this scenario, an arriving customer joins the queue with probability q with $q \in [0, 1]$. If all customers apply strategy q, the arrival process becomes the geometric process with the parameter pq. Note that when $q = 1$ or $q = 0$, the strategy becomes pure "always join" or "always balk" case. Such a decision rule is called q-joining strategy.

To analyze the equilibrium strategy, we first derive the expected sojourn time (also called system time). Let L be the queue length (including the customer in service), I be the server state($I \in \{0, 1\}$, 1: busy, 0: vacation), S be the sojourn time of the joining customer, and p_i be the probability of the server state, $i \in \{0, 1\}$. From the Little's law and the results in Tian and Zhang [14], we have

$$p_1 = pqE[B], \qquad E[L] = pqE[S]. \tag{3}$$

The probability that an arriving customer enters to the system and finds that the server is in state $i(i \in \{0, 1\})$ is

$$\frac{pqp_i}{pqp_0 + pqp_1} = p_i.$$

If the customer finds that the server is on vacation, his sojourn time has two parts: the residual vacation time and $L + 1$ service time. If he finds that the server is busy, his sojourn time is the addition of one residual service time and L service time. Thus, we obtain

$$E[S] = p_0(E[R_V] + (E[L] + 1)E[B]) + p_1(E[R_B] + E[L]E[B]). \tag{4}$$

It follows from $p_0 + p_1 = 1$ that the following result can be established.

Lemma 1. In a Geo/G/1/MV queue with NI, if an arriving customer applies the q joining strategy, his expected sojourn time is

$$E[S] = E[R_V] + E[B] + \frac{pqE[B]}{1 - pqE[B]}E[R_B]. \tag{5}$$

For an arriving customer, if he chooses to balk, his payoff is 0, and if he chooses to enter, his payoff function, denoted by $R_e(q)$ is given by

$$R_e(q) = K - CE[S] = K - C\left(E[R_V] + E[B] + \frac{pqE[B]}{1 - pqE[B]}E[R_B]\right). \tag{6}$$

Evidently, if $R_e(q) > 0$, the customer decides to enter; if $R_e(q) < 0$; he decides to balk; if $R_e(q) = 0$, he is indifferent between entering and balking. Because the equation $R_e(q)$ is a monotone decreasing function for q, it has the unique solution q_e^* to (6):

$$q_e^* = \frac{1}{pE[B]}\left(1 - \frac{E[R_B]}{\frac{K}{C} - E[R_V] - E[B] + E[R_B]}\right). \tag{7}$$

and

$$\begin{cases} q_e^* \leq 0, & \frac{K}{C} \leq E[R_V] + E[B], \\ q_e^* \in (0,1), & E[R_V] + E[B] < \frac{K}{C} < E[R_V] + E[B] + \frac{pE[B]}{1-pE[B]}E[R_B], \\ q_e^* \geq 1, & E[R_V] + E[B] + \frac{pE[B]}{1-pE[B]}E[R_B] \leq \frac{K}{C}. \end{cases}$$

Based on the conditions for different value ranges of q_e^*, we can analyze the equilibrium joining strategies of customers. There are three cases regarding the customer strategic responses.

Case 1. If

$$\frac{K}{C} \leq E[R_V] + E[B],$$

for all $\forall q \in [0,1]$, then $R_e(q) \leq 0$. Thus, ”never entering (i.e. entering with probability zero)” is the customer’s unique equilibrium strategy.

Case 2. If

$$E[R_V] + E[B] < \frac{K}{C} < E[R_V] + E[B] + \frac{pE[B]}{1-pE[B]}E[R_B],$$

it follows from the monotonicity of $R_e(q)$ that $R_e(q_e^*) = 0$, and when $q < q_e^*$, $R_e(q) > 0$; when $q > q_e^*$, $R_e(q) < 0$. Based on Nash equilibrium, ”entering with probability q_e^*” is the customer’s unique equilibrium strategy.

Case 3. If

$$E[R_V] + E[B] + \frac{pE[B]}{1-pE[B]}E[R_B] \leq \frac{K}{C},$$

for all $\forall q \in [0,1]$, $R_e(q) \geq 0$. Thus, ”always entering (i.e. entering with probability 1)” is the customer’s unique equilibrium strategy.

We can summarize the results above as follows:

Theorem 1. In a Geo/G/1/MV queue with NI, under the conditions (1) and (2), “entering with probability q_e” is the customer’s unique equilibrium strategy, where q_e is

$$q_e = \begin{cases} q_e^*, & E[R_V] + E[B] < \frac{K}{C} < E[R_V] + E[B] + \alpha E[R_B] \\ 1, & E[R_V] + E[B] + \alpha E[R_B] \leq \frac{K}{C} \end{cases} \tag{8}$$

where q_e^* is given in equation (7) and $\alpha = \frac{pE[B]}{1-pE[B]}$.

2.3. Social optimal analysis

From the queueing manager's perspective, maximizing the social welfare may become the system objective. To determine the joining probability for maximizing social welfare, we first write down the social welfare as a function of q. That is

$$R_s(q) = pqK - CE[L] = pq\left(K - C\bigg(E[R_V] + E[B] + \frac{pE[B]}{1-pE[B]}E[R_B]\bigg)\right). \quad (9)$$

Evidently, the social welfare function $R_s(q)$ has two parameters (K, C).

Solving the equation $dR_s(q)/dq = 0$, we can obtain the social optimal

$$q_s^* = \frac{1}{pE[B]}\left(1 - \sqrt{\frac{E[R_B]}{\frac{K}{C} - E[R_V] - E[B] + E[R_B]}}\right). \quad (10)$$

Because for $\forall q \in [0, 1]$, $pqE[B] < 1$, thus $R_s^{''}(q) < 0, \forall q \in [0, 1]$, so the equation $R_s(q)$ is a concave function and $q = q_s^*$ is global minimum. Clearly,

$$\begin{cases} q_s^* \leq 0, & \frac{K}{C} \leq E[R_V] + E[B], \\ q_s^* \in (0, 1), & E[R_V] + E[B] < \frac{K}{C} < E[R_V] + E[B] + \frac{pE[B]}{1-pE[B]}(1 + \frac{1}{1-pE[B]})E[R_B], \\ q_s^* \geq 1, & E[R_V] + E[B] + \frac{pE[B]}{1-pE[B]}(1 + \frac{1}{1-pE[B]})E[R_B] \leq \frac{K}{C}. \end{cases}$$

Similar to the customer's equilibrium decision, there are three cases regarding the social welfare maximization strategy.

Case 1. If

$$\frac{K}{C} \leq E[R_V] + E[B],$$

for $\forall q \in [0, 1]$, then $R_s'(q) \leq 0$. Because the equation (9) is a decreasing function for q, $\forall q \in [0, 1]$, $R_s(0) \geq R_s(q)$. Thus, "entering with probability 0" is the unique social optimal strategy.

Case 2. If

$$E[R_V] + E[B] < \frac{K}{C} < E[R_V] + E[B] + \frac{pE[B]}{1-pE[B]}\left(1 + \frac{1}{1-pE[B]}\right)E[R_B],$$

for $\forall q \in [0, 1]$, $R_s^{''}(q) < 0$ and $R_s'(q_s^*) = 0$. When $q < q_s^*$, $R_s'(q) > 0$; when $q > q_s^*$, $R_s'(q) < 0$, in other word, when $q \in [0, q_s^*)$, $R_s(q)$ is monotonically increasing for q, and when $q \in (q_s^*, 1]$, $R_s(q)$ is monotonically decreasing for q. It is easily to obtain that $R_s(q)$ achieves to the maximum value at the point $q = q_s^*$. Based on Nash equilibrium, "entering with probability q_s^*" is the unique social optimal strategy.

Case 3. If

$$E[R_V] + E[B] + \frac{pE[B]}{1-pE[B]}\left(1 + \frac{1}{1-pE[B]}\right)E[R_B] \leq \frac{K}{C},$$

for $\forall q \in [0,1]$, $R_s'(q) \geq 0$. Because the equation (9) is an increasing function for q, $\forall q \in [0,1]$, $R_s(1) \geq R_s(q)$. Thus, "entering with probability 1" is the unique social optimal strategy.

Summarizing these cases leads to the following theorem.

Theorem 2. In a Geo/G/1/MV queue with NI, under the condition of equation (1) and (2), "to enter with probability q_s" is the customer's unique social equilibrium strategy, where q_s is

$$q_s = \begin{cases} q_s^*, & E[R_V] + E[B] < \frac{K}{C} < E[R_V] + E[B] + \beta E[R_B] \\ 1, & E[R_V] + E[B] + \beta E[R_B] \leq \frac{K}{C} \end{cases} \tag{11}$$

and q_s^* is given in the equation (10), and

$$\beta = \frac{pE[B]}{1 - pE[B]}\left(1 + \frac{1}{1 - pE[B]}\right).$$

Based on Theorems 1 and 2, we obtain the following results.

Theorem 3. In a Geo/G/1/MV queue with NI, $q_s \leq q_e$, and the mixture of the individual equilibrium strategy and social equilibrium strategy is

$$(q_e, q_s) = \begin{cases} (q_e^*, q_s^*), & E[R_V] + E[B] < \frac{K}{C} < E[R_V] + E[B] + \alpha E[R_B] \\ (1, q_s^*), & E[R_V] + E[B] + \alpha E[R_B] < \frac{K}{C} < E[R_V] + E[B] + \beta E[R_B] \\ (1, 1), & E[R_V] + E[B] + \beta E[R_B] \leq \frac{K}{C} \end{cases} \tag{12}$$

In fact, when customers make their joining decisions to maximize their own utilities, they always ignore the negative effect on others, called negative externality. However, when we consider maximizing the social welfare, such a negative externality is taken into account. Therefore, the joining probability for the socially optimal strategy will be smaller than that for the individual equilibrium joining probability.

3. The PI case

3.1. Model formulation

In the PI case, an arriving customer knows the server state of either on duty or on vacation. The decision rule can be represented by two number (d_0, d_1), where d_0 (d_1) represents the customer's decision when the server is on vacation (on duty) with $d_0 = 1$ $(d_1 = 1)$ being "joining" and $d_0 = 0$ $(d_1 = 0)$ being "balking". Then, there are four pure strategies for customers: (a) always balking (0,0), i.e., balking regardless of server state; (b) balking only when the server is on vacation (0,1); (c) balking only when the server is on duty (1,0); and (d) always joining (1,1) i.e., joining regardless of server state. A randomized (mixed) strategy can be denoted by (q_0, q_1), where q_i is the probability that the customer decides to join when the server's state is $i, i \in \{0,1\}$. Similar to the previous section, we use the mean value analysis for the PI case.

Here, we define L_i as the queue length when the server is in state where $i(i \in \{0,1\})$ and S_i as the sojourn time of the customer joining at server's state i where $(i \in \{0,1\})$. Further, we define p_i as the probability that an arrival customer finds that the server is in state i where $(i \in \{0,1\})$. Similarly, to ensure the stability of the system, we assume

$$pq_1E[B] < 1. \tag{13}$$

Donate by $\overline{p}$ the effective arrival rate of customers.

$$\overline{p} = p(q_0p_0 + q_1p_1). \tag{14}$$

According to Little law, we have

$$p_1 = \overline{p}E[B], \qquad E[L] = \overline{p}E[S]. \tag{15}$$

Evidently, $E[L]$ can also be expressed as $E[L] = p_0E[L_0] + p_1E[L_1]$, and we know from (13), (14) and $p_0 + p_1 = 1$ that

$$\begin{array}{ll} p_0 = \frac{1-pq_1E[B]}{1-p(q_1-q_0)E[B]}, & p_1 = \frac{pq_0E[B]}{1-p(q_1-q_0)E[B]}, \\ \overline{p} = \frac{pq_0}{1-p(q_1-q_0)E[B]}. & \end{array} \tag{16}$$

3.2. Equilibrium analysis under the PI scenario

Mark a customer who chooses to join the system, and donate by π_i the probability that this marked customer enters the system by knowing that the server's state is i, then we have

$$\pi_i = \frac{pq_ip_i}{pq_0p_0 + pq_1p_1}, i \in \{0,1\}.$$

From (14), we obtain

$$\pi_i = \frac{pq_ip_i}{\overline{p}}, i \in \{0,1\}.$$

It follows from (16) that

$$\pi_0 = 1 - pq_1E[B], \qquad \pi_1 = pq_1E[B].$$

Thus, we have

$$E[S] = (1 - pq_1E[B])E[S_0] + pq_1E[B]E[S_1]. \tag{17}$$

If a customer finds that the server is on vacation, then his sojourn time is the remaining vacation time plus $L_0 + 1$ service time. According to PASTA property, upon the certain customer arrival, the distribution of the queue length and the distribution of system length L_0 are the same. Notice that when the server is on vacation, the process of customers arrival follows a geometric distribution with parameter pq_0. Then the next equation holds:

$$E[S_0] = E[R_V] + (E[L_0] + 1)E[B]. \tag{18}$$

If the customer finds that the server is busy, his sojourn time is the remaining service time of the customer being served plus L_1 service time. Thus, we have

$$E[S_1] = E[R_B] + E[L_1]E[B]. \tag{19}$$

When the server is on vacation, customer arrivals follow a geometric distribution with parameter pq_0. Because the system is in steady state, according to Little law, we have

$$E[L_0] = pq_0 E[R_V]. \tag{20}$$

From (15), (19) and (20), we have

Lemma 2. In a Geo/G/1/MV queueing model with PI, under the assumption that all customers adopt the equilibrium strategy (q_0, q_1), the expected sojourn times of customers joining at the server's vacation state and on-duty state are

$$\begin{aligned} E[S_0] &= E[R_V] + (pq_0 E[R_V] + 1)E[B], \\ E[S_1] &= (pq_0 E[R_V] + 1)E[B] + \tfrac{E[R_B]}{1-pq_1 E[B]}, \end{aligned} \tag{21}$$

, respectively. The average queue length is given by

$$E[L] = pq_0 \left(E[R_V] + \left(\frac{pq_1 E[R_B]}{1 - pq_1 E[B]} + 1 \right) \frac{E[B]}{1 - p(q_1 - q_0)E[B]} \right). \tag{22}$$

Note that from (21) the expected reward of customer joining a vacation state, denoted by $R_e(0; q_0)$, is

$$R_e(0; q_0) = K - CE[S_0] = K - C(E[R_V] + (pq_0 E[R_V] + 1)E[B]), \tag{23}$$

which does not depend on q_1. Because both q_0 and q_1 are in the expression of $E[S_1]$, we denote the expected reward of the customer joining a "server on duty" state by $R_e(1; q_0, q_1)$, which is given by

$$R_e(1; q_0, q_1) = K - CE[S_1] = K - C\left((pq_0 E[R_V] + 1)E[B] + \frac{E[R_B]}{1 - pq_1 E[B]} \right). \tag{24}$$

Remark 1. The equilibrium strategies (q_0, q_1) can be determined by the iterative algorithm as follows.

We will compute the values of q_0 and q_1 iteratively. First, starting with (23), if $R_e(0; q_0) > 0$, the customer chooses to join the system; if $R_e(0; q_0) = 0$, he is indifferent between joining and balking; and if $R_e(0; q_0) < 0$, the customer chooses to balk. It's obvious that (23) strictly decreases with q_0, thus, there exists only one zero solution $q_e^*(0)$:

$$q_e^*(0) = \frac{1}{pE[R_V]E[B]} \left(\frac{K}{C} - E[R_V] - E[B] \right). \tag{25}$$

and

$$\begin{cases} q_e^*(0) \leq 0, & \frac{K}{C} \leq E[R_V] + E[B], \\ q_e^*(0) \in (0,1), & E[R_V] + E[B] < \frac{K}{C} < E[R_V] + (pE[R_V]+1)E[B], \\ q_e^*(0) \geq 1, & E[R_V] + (pE[R_V]+1)E[B] \leq \frac{K}{C}. \end{cases}$$

Lemma 3. In a Geo/G/1/MV queueing model with PI, if (1) and (2) hold, and the server is on vacation, then "joining with probability $q_e(0)$" is the unique equilibrium strategy. Denote $q_e(0)$ by

$$q_e(0) = \begin{cases} q_e^*(0) & E[R_V] + E[B] < \frac{K}{C} < E[R_V] + \gamma E[B], \\ 1 & E[R_V] + \gamma E[B] \leq \frac{K}{C}. \end{cases} \tag{26}$$

where $q_e^*(0)$ is given by (25) and $\gamma = pE[R_V] + 1$.

After computing $q_e(0)$, we will compute the value of $q_e(1)$ from (24). To simplify the question, donate $R_e(1; q_e(0), q_1)$ by $\overline{R}_e(1; q_1)$, where $q_e(0)$ is given in (26). When the server is in a busy period, $q_e(0)$ is known, and if $\overline{R}_e(1; q_1) > 0$, he chooses to join; if $\overline{R}_e(1; q_1) < 0$, he chooses to balk; and if $\overline{R}_e(1; q_1) = 0$, he is indifferent. Due to the two values of $q_e(0)$, we consider two cases.

Case 1. When $E[R_V] + E[B] < \frac{K}{C} < E[R_V] + (pE[R_V]+1)E[B]$, $q_e(0) = q_e^*(0)$. Substituting (25) into (26), we have

$$\overline{R}_e(1; q_1) = C\left(E[R_V] - \frac{E[R_B]}{1 - pq_1E[B]}\right).$$

It's obvious that $\overline{R}_e(1; q_1)$ is strictly decreasing with q_1, then the equation above has only one zero solution $q_e^*(1)$,

$$q_e^*(1) = \frac{1}{pE[B]}\left(1 - \frac{E[R_B]}{E[R_V]}\right)$$

and

$$\begin{cases} q_e^*(1) \leq 0, & E[R_V] \leq E[R_B], \\ q_e^*(1) \in (0,1), & E[R_B] < E[R_V] < \frac{E[R_B]}{1-pE[B]}, \\ q_e^*(1) \geq 1, & \frac{E[R_B]}{1-pE[B]} \leq E[R_V]. \end{cases}$$

Lemma 4. In a Geo/G/1/MV queueing model with PI, if equation (1) and (2) hold, and (K, C) satisfies

$$E[R_V] + E[B] < \frac{K}{C} < E[R_V] + (pE[R_V]+1)E[B],$$

that means $q_e(0) = q_e^*(0)$, and the server is in the busy state, then "join with probability $q_e(1)$" is the only equilibrium strategy, where $q_e(1)$ is

$$q_e(1) = \begin{cases} 0, & E[R_V] \leq E[R_B], \\ q_e^*(1), & E[R_B] < E[R_V] < \delta, \\ 1, & \delta \leq E[R_V]. \end{cases} \tag{27}$$

where

$$\delta = \frac{E[R_B]}{1 - pE[B]}.$$

Case 2. When $E[R_V] + (pE[R_V] + 1)E[B] \leq \frac{K}{C}$, $q_e(0) = 1$, then substituting $q_e(0) = 1$ into (24), we have

$$\overline{R}_e(1; q_1) = K - C\left((pE[R_V] + 1)E[B] + \frac{E[R_B]}{1 - pq_1 E[B]}\right).$$

Similarly, because $\overline{R}_e(1; q_1)$ is strictly decreasing with q_1, there exists only one zero solution $\overline{q}_e^*(1)$,

$$\overline{q}_e^*(1) = \frac{1}{pE[B]}\left(1 - \frac{E[R_B]}{\frac{K}{C} - (pE[R_V] + 1)E[B]}\right). \tag{28}$$

Lemma 5. In a Geo/G/1/MV queueing model with PI, if (1) and (2) hold, and (K, C) satisfies

$$E[R_V] + (pE[R_V] + 1)E[B] \leq \frac{K}{C},$$

and the server is in the busy state, then "joining with probability $q_e(1)$" is the only equilibrium strategy, where $q_e(1)$ is

$$q_e(1) = \begin{cases} 0, & E[R_V] \leq E[R_B] \text{ and } \frac{K}{C} \leq E[R_B] + \gamma E[B], \\ \overline{q}_e^*(1), & \text{or } E[R_V] \leq E[R_B] \text{ and } E[R_B] + \gamma E[B] \\ & < \frac{K}{C} < \delta + \gamma E[B], \\ & \text{or } E[R_B] < E[R_V] < \delta \text{ and } \frac{K}{C} < \delta + \gamma E[B], \\ 1, & \text{or } \delta \leq E[R_V], \\ & \text{or } E[R_V] < \delta \text{ and } \delta + \gamma E[B] \leq \frac{K}{C}. \end{cases} \tag{29}$$

and $\overline{q}_e^*(1)$ is given by (28), $\gamma = pE[R_V] + 1$, and $\delta = \frac{E[R_B]}{1-pE[B]}$.

Summarizing the results in Lemma 3, 4 and 5, we have

Theorem 4. In a Geo/G/1/MV queueing model with PI, if (1) and (2) hold, there exists only one equilibrium strategy $(q_e(0), q_e(1))$, if the server is on vacation, the customer joins with probability $q_e(0)$; if the server is busy, the customer joins with probability $q_e(1)$, where $(q_e(0), q_e(1))$ are given in the following cases:

I. If $E[R_V] \leq E[R_B]$ holds,

$$(q_e(0), q_e(1)) = \begin{cases} (q_e^*(0), 0), & E[R_V] + E[B] < \frac{K}{C} < E[R_V] + \gamma E[B], \\ (1, 0), & E[R_V] + \gamma E[B] \leq \frac{K}{C} \leq E[R_B] + \gamma E[B], \\ (1, \overline{q}_e^*(1)), & E[R_B] + \gamma E[B] < \frac{K}{C} < \delta + \gamma E[B], \\ (1, 1), & \delta + \gamma E[B] \leq \frac{K}{C}. \end{cases}$$

II. If $E[R_B] < E[R_V] < \delta$ holds,

$$(q_e(0), q_e(1)) = \begin{cases} (q_e^*(0), q_e^*(1)), & E[R_V] + E[B] < \frac{K}{C} < E[R_V] + \gamma E[B], \\ (1, \overline{q}_e^*(1)), & E[R_V] + \gamma E[B] \leq \frac{K}{C} < \delta + \gamma E[B], \\ (1, 1), & \delta + \gamma E[B] \leq \frac{K}{C}. \end{cases}$$

III. If $\delta \leq E[R_V]$ holds,

$$(q_e(0), q_e(1)) = \begin{cases} (q_e^*(0), 1), & E[R_V] + E[B] < \frac{K}{C} < E[R_V] + \gamma E[B], \\ (1, 1), & E[R_V] + \gamma E[B] \leq \frac{K}{C}. \end{cases}$$

where $\gamma = pE[R_V] + 1$, and $\delta = \frac{E[R_B]}{1-pE[B]}$.

Table 1. The equilibrium strategy $(q_e(0), q_e(1))$ in the Geo/G/1/MV queue with PI

I	$(q_e(0), q_e(1))$	II	$(q_e(0), q_e(1))$	III	$(q_e(0), q_e(1))$
$\frac{K}{C} \in (\kappa_1, \kappa_2)$	(+,0)	$\frac{K}{C} \in (\kappa_1, \kappa_2)$	(+,+)	$\frac{K}{C} \in (\kappa_1, \kappa_2)$	(+,1)
$\frac{K}{C} \in (\kappa_2, \kappa_3)$	(1,0)	$\frac{K}{C} \in (\kappa_2, \kappa_4)$	(1,+)	$\frac{K}{C} \in (\kappa_2, +\infty)$	(1,1)
$\frac{K}{C} \in (\kappa_3, \kappa_4)$	(1,+)	$\frac{K}{C} \in (\kappa_4, +\infty)$	(1,1)		
$\frac{K}{C} \in (\kappa_4, +\infty)$	(1,1)				

Table 1 shows the equilibrium strategy $(q_e(0), q_e(1))$ in Theorem 4 with different K/C ratios. Note that κ_i in Table 1 can be defined as follows:

$$\kappa_1 = E[R_V] + E[B],$$
$$\kappa_2 = E[R_V] + (pE[R_V] + 1)E[B],$$
$$\kappa_3 = E[R_B] + (pE[R_V] + 1)E[B],$$
$$\kappa_4 = \frac{E[R_B]}{1 - pE[B]} + (pE[R_V] + 1)E[B].$$

The symbol "+" means the joining probability in the interval (0; 1), for example, (+,1) is equal to a certain strategy $(q_e(0), q_e(1))$, where $0 < q_e(0) < 1, q_e(1) = 1$.

Remark 2. Theorem 4 reveals some customers' joining behaviors under the PI scenario. Intuitively, we may think that customers prefer joining at the server busy state to joining at the server vacation state, i.e., $q_e(0) \leq q_e(1)$. However, such an intuition is only true under certain conditions such as case III. More complex relations between the two joining probabilities also exist as follows:

1. In Case I where $E[R_V] < E[R_B]$, customers prefer joining at server busy state to joining at server vacation state. That is $q_e(0) \geq q_e(1)$.
2. Case II is a grey zone between Case I and Case III, and the relation between $q_e(0)$ and $q_e(1)$ needs additional conditions To be specific, when (K, C) satisfies $K/C < E[R_V] + (pE[R_V] + 1)E[B]$, if $E[R_V] + E[B] - E[R_B] \leq K/C$, then $q_e^*(0) = \frac{1}{pE[R_V]E[B]}\left(\frac{K}{C} - E[R_V] - E[B]\right) \geq \frac{1}{pE[B]}\left(1 - \frac{E[R_B]}{E[R_V]}\right) = q_e^*(1).$, otherwise, $q_e(0) < q_e(1)$; and when (K, C) satisfies $E[R_V] + (pE[R_V] + 1)E[B] \leq K/C$, then $q_e(0) \geq q_e(1)$.

It is worth pointing that the social welfare optimal strategy under the PI scenario is harder to analyze using the current approach. Under the equilibrium strategy $(q_e(0), q_e(1))$, the social welfare can be written as

$$R_w = p\pi_1 q_e(1)K - CE[L_0] + p\pi_0 q_e(0)K - CE[L_1]. \tag{30}$$

Clearly, the social welfare optimal strategy is the mixed strategy $(q_s(0), q_s(1))$ that maximizes $R_s(q_s(0), q_s(1))$, which is given by

$$R_s(q_s(0), q_s(1)) = p(\pi_{s1} q_s(1) + \pi_{s0} q_s(0))K - CE[L]. \tag{31}$$

Here, we cannot derive the explicit expressions for $\pi_{s0} = 1 - pq_s(1)E[B]$ and $\pi_{s1} = pq_s(1)E[B]$. Such a complexity prevents us from analyzing the social welfare optimal joining strategy in PI case.

4. Numerical Illustrations

In this section, we present numerical examples for the two information cases. Here, we assume that the holding cost is $C = 1$ per time unit in the queue.

Example 1: q_e and q_s as functions of system parameters for the NI case
Consider q_e and q_s as functions of one of the system parameters(K, p, $E[B]$, $E[R_B]$ and $E[R_V]$) while keeping other parameters fixed. Figures 1 and 2 show how customer equilibrium strategy q_e and social optional strategy q_s change with K, p, $E[R_B]$, and $E[R_V]$ respectively, with the corresponding constant parameter values given as

$(p, E[B], E[R_B], E[R_V]) = (0.5, 1.0, 4.0, 4.6)$(Figure 1(a)),
$(K, E[B], E[R_B], E[R_V]) = (7.0, 1.0, 4.0, 4.6)$(Figure 1(b)),
$(K, p, E[B], E[R_V]) = (7.0, 0.2, 3.0, 2.0)$(Figure 2(a)),
$(K, p, E[B], E[R_B]) = (7.0, 0.2, 3.0, 2.0)$(Figure 2(b)), respectively.

As shown in Figure 1, the values of customer equilibrium strategy q_e and social optimal strategy q_s increase with the service reward value K, and decrease with the arrival probability p (arrival rate). While the former relation is intuitive, the latter indicates the effective arrival rate to the queueing system is adjusted to an appropriate level when the arrival rate increases. of p.

Figure 2 shows q_e and q_s decrease with $E[R_B]$ and $E[R_V]$. Such a relation reflects the fact that the longer residual service time or residual vacation time will lead to more congestion system which in turns reduces these joining probabilities. It follows from these figures that the relation $q_s \leq q_e$ is true in these numerical examples.

Example 2: q_e in NI case and $(q_e(0), q_e(1))$ in PI case as functions of system parameters
Consider q_e and $(q_e(0), q_e(1))$ as functions of one of the system parameters(K, p, $E[B]$, $E[R_B]$ and $E[R_V]$) while keeping other parameters fixed.

Figures 1 and 2 show how customer equilibrium strategy q_e and $(q_e(0), q_e(1))$ change with K, p, $E[R_B]$, and $E[R_V]$ respectively, with the corresponding constant parameter values given as

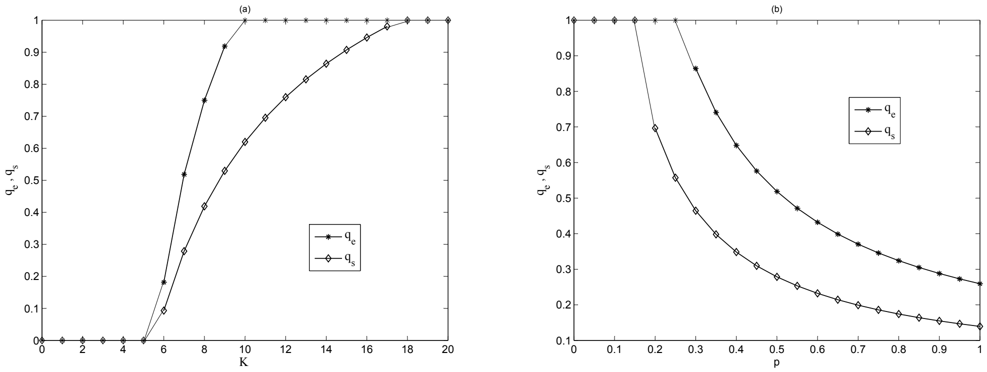

Figure 1. The Curve of q_e and q_s versus K and p

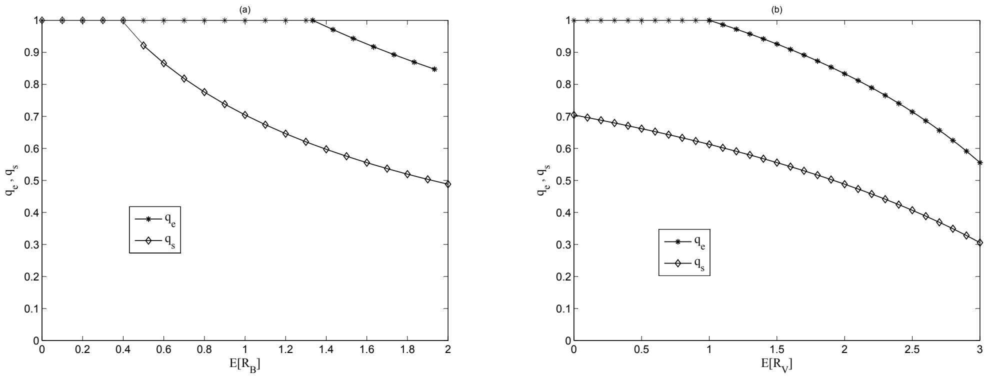

Figure 2. The Curve of q_e and q_s versus $E[R_B]$ and $E[R_V]$

$(p, E[B], E[R_B], E[R_V]) = (0.5, 1.0, 4.0, 4.6)$(Figure 3(a)),
$(K, E[B], E[R_B], E[R_V]) = (7.0, 1.0, 4.0, 4.6)$(Figure 3(b)),
$(K, p, E[B], E[R_V]) = (7.0, 0.2, 3.0, 2.0)$(Figure 4(a)),
$(K, p, E[B], E[R_B]) = (7.0, 0.2, 3.0, 2.0)$(Figure 4(b)).

Figures 3 and 4 show the relations among q_e, $q_e(0)$ and $q_e(1)$ curves can be one of three cases : i) q_e in NI case stays in between $q_e(0)$ and $q_e(1)$ in PI case; ii) q_e coincides with either $q_e(0)$ or $q_e(1)$; and iii) They all coincide. Such an observation indicates that the information level may affect the customers' equilibrium strategy.

In Figure 3, the customers' equilibrium joining probabilities depend on K and p in a similar way as in Figure 1.

In Figure 4, while q_e in the NI case always decreases with $E[R_B]$ and $E[R_V]$, $q_e(0)$ and $q_e(1)$ in the PI case seem to be more complex. Although $q_e(0)$ is a decreasing function of $E[R_V]$, it is independent of $E[R_B]$. More interestingly, while $q_e(1)$ is a decreasing function of $E[R_B]$, it is not monotonic function of $E[R_V]$. That is, it is decreasing first, and then increasing as shown in Figure 4 (b).

Remark 3. It is worth noting that q_e $q_e(0)$ and $q_e(1)$ intersect at the same point in Figure 4 (b), that is between 0 and 1. This happens only when

$$E[R_B] < E[R_V] < \frac{E[R_B]}{1 - pE[B]}.$$

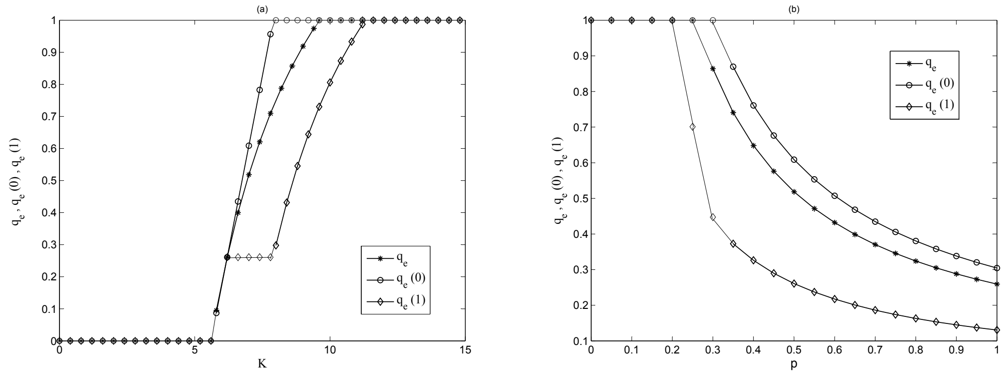

Figure 3. The Curve of q_e versus K and p

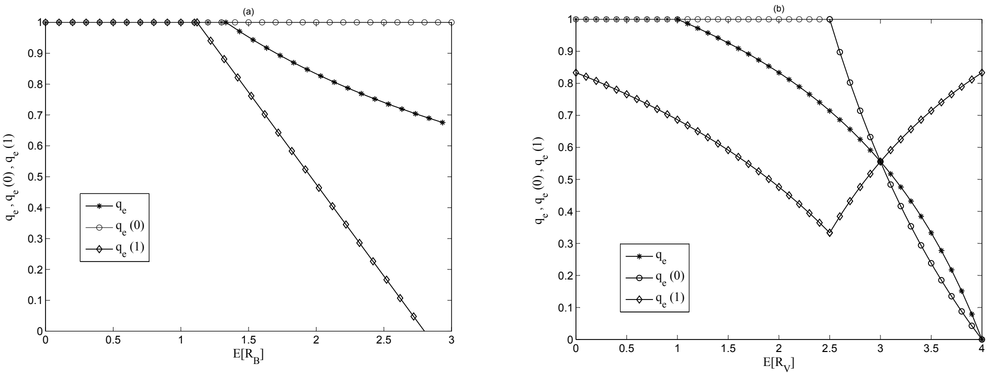

Figure 4. The Curve of q_e versus $E[R_B]$ and $E[R_V]$

and

$$E[R_V] + E[B] < \frac{K}{C} < E[R_V] + (pE[R_V] + 1)E[B].$$

holds at the same time. In addition, it follows from the expressions of q_e, $q_e(0)$ and $q_e(1)$ that $q_e = q_e(0) = q_e(1)$ holds if and only if $K/C = 2E[R_V] + E[B] - E[R_B]$.

Example 3: Social welfare as functions of K and p for the NI case
Now we plot the welfare $R_s(q)$; $q \in \{q_e, q_s\}$ as the functions of K and p, respectively. Consider $E[B] = 0.4, E[R_B] = 3, E[R_V] = 0.4, p = 2.0$ (Figure 5(a)), and $K = 15$ (Figure 5(b)).

In Figure 5(a), the difference between $R_s(q_s)$ and $R_s(q_e)$ represents the loss of social welfare due to customers' following self-interest equilibrium strategy. In Figure 5(b), because of large K, q_e and q_s are equal to 1, and two curves of $R_s(q_e)$ and $R_s(q_s)$ coincide.

5. Conclusions

Using the mean value analysis, a relative simple approach, this note studies Geo/G/1/MV queueing models with customers having choices to join or balk a queue under two information scenarios: the no information and partial information cases. The numerical analysis is presented to gain some managerial insights. For example, it has been observed that the

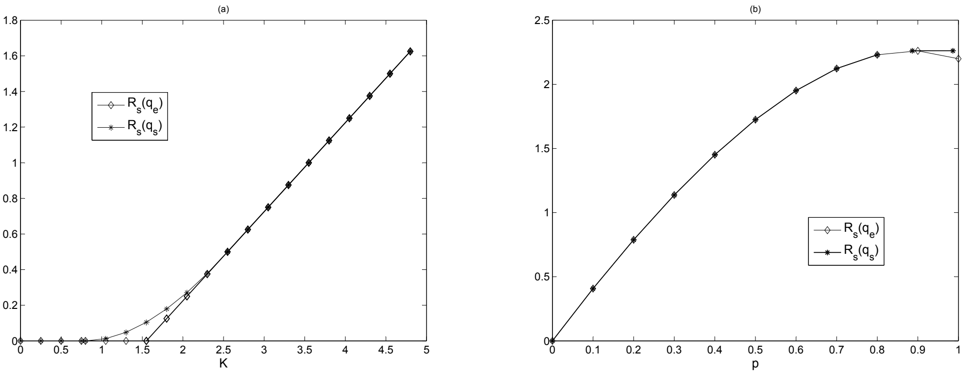

Figure 5. Social Reward per time versus K and p if Choosing q_e or q_s

customer joining probabilities can be affected by the information about the server state in a complex way. The numerical results also confirm that in the no information case, the customer equilibrium joining probability is greater than the social welfare optimal joining probability.

However, this work has some limitations. First, we could not work out the explicit results for the social optimal joining strategy under the partial information scenario. Second, we did not consider the full information scenario where the actual queue length is disclosed to arriving customers. Finally, the competition and cooperation among service providers in a queueing setting with customer joining strategy can be interesting issues to be addressed. These topics can be good directions for future research.

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