Queueing Models and
Service Management
Vol. 1, No. 1, page 095-129, 2018

QMSM

A Kernel Method for Exact Tail Asymptotics-Random Walks in the Quarter Plane

(In memory of Dr. Philippe Flajolet)

Hui Li[1] and Yiqiang Q. Zhao[2,*]

[1]Department of Mathematics
Mount Saint Vincent University
Halifax, NS Canada B3M 2J6

[2]School of Mathematics and Statistics
Carleton University
Ottawa, ON Canada K1S 5B6

(Received March 2017; accepted November 2017)

Abstract: In this paper, we propose a kernel method for exact tail asymptotics of a random walk to neighborhoods in the quarter plane, which is a general model for two-dimensional queueing systems with many important applications in various areas including service management. This is a two-dimensional method, which does not require a determination of the unknown generating function(s). Instead, in terms of the asymptotic analysis and a Tauberian-like theorem, we show that the information about the location of the dominant singularity or singularities and the detailed asymptotic property of the unknown function at a dominant singularity is sufficient for the exact tail asymptotic behaviour for the marginal distributions and also for joint probabilities along a coordinate direction. We provide all details, not only for a "typical" case, the case with a single dominant singularity for an unknown generating function, but also for all non-typical cases which have not been studied before. A total of four types of exact tail asymptotics are found for the typical case, which have been reported in the literature. We also show that on the circle of convergence, an unknown generating function could have two dominant singularities instead of one, which can lead to a new periodic phenomena. Examples are illustrated by using this kernel method. This paper can be considered as a systematic summary and extension of existing ideas, which also contains new and interesting research results.

Keywords: Exact tail asymptotics, generating functions, kernel methods, light tail, queueing systems, random walks in the quarter plane, singularity analysis, stationary distributions.

1. Introduction

Two-dimensional discrete random walks in the quarter plane are classical models, that could be either probabilistic or combinatorial. The probabilistic random walk in the quarter plane is a general model for two-dimensional queusing systems. For example, the well-known Jackson open network, join-the-shortest-queue system, priority queueing system, two-demand queues, systems with cross-trained servers, 2×2 switches, processor sharing systems, etc. can all be modelled as a random walk in the quarter plane. These queueing systems, as well as many others, represent important applications in many fields, including the ones in service management. Studying these models is important and often fundamental for both theoretical and applied purposes. For a stable probabilistic model, it is of significant interest to study its stationary probabilities. However, only for very limited special cases, a closed-form solution is available for the stationary probability distribution. This adds value to studying tail asymptotic properties in stationary probabilities, since performance bounds and approximations can often be developed from the tail asymptotic property. The focus of this paper is to characterize exact tail asymptotics. Specifically, we propose a kernel method to systematically study the exact tail behaviour for the stationary probability distribution of the random walk in the quarter plane.

* Corresponding author
Email : zhao@math.carleton.ca

The kernel method proposed here is an extension of the classical kernel method, first introduced by Knuth [32] and later developed as the kernel method by Banderier et al.[3]. The standard kernel method deals with the case of a functional equation of the fundamental form $K(x,y)F(x,y) = A(x,y)G(x) + B(x,y)$, where $F(x,y)$ and $G(x)$ are unknown functions. The key idea in the kernel method is to find a branch $y = y_0(x)$, such that, at $(x, y_0(x))$, the kernel function is zero, or $K(x, y_0(x)) = 0$. When analytically substituting this branch into the right hand side of the fundamental form, we then have $G(x) = -B(x, y_0(x)) / A(x, y_0(x))$, and hence,

$$F(x,y) = \frac{-A(x,y)B(x,y_0(x))/A(x,y_0(x)) + B(x,y)}{K(x,y)}.$$

However, applying the above idea to the fundamental form of a two-dimensional random walk does not immediately lead to a determination of the generating function $P(x,y)$. Instead, it provides a relationship between two unknown generating functions $\pi_1(x)$ and $\pi_2(y)$, referred to as the generating functions for the boundary probabilities. This is the key challenge in the analysis of using the kernel method. Therefore, a good understanding on the interlace of these two functions is crucial.

Following the early research by Malyshev [43, 44], the algebraic method targeting on expressing the unknwon generating functions was further systematically updated in Fayolle et al.[14] based on the study of the kernel equation. The authors indicated in their book that: "Even if asymptotic problems were not mentioned in this book, they have many applications and are mostly interesting for higher dimensions." The proposed kernel method in this paper is a continuation of the study in [14]. Research on tail asymptotics for various models following the method (determination of the unknown generating function(s) first) of [14] or other closely related methods can be found in Flatto and McKean [17], Fayolle and Iasnogorodski [12], Fayolle et al. [13], Cohen and Boxma [8], Flatto and Hahn [18], Flatto [19], Wright [61], Kurkova and Suhov [34], Bousquet-Melou [6], Morrison [54], Li and Zhao [38, 39], Guillemin and Leeuwaarden [23], and Li et al. [36].

Different from the work mentioned above, which requires characterizing or expressing the unknown generating function, such as a closed-form solution or an integral expression through boundary value problems, the proposed kernel method only requires the information about the dominant singularities of the unknown function, including the location and detailed asymptotic property at the dominant singularities. Because of this, our method makes it possible to systematically deal with all random walks instead of a model based treatment. In a recent research, Li and Zhao [40] applied this method to a specific model, and Li et al.[36] to the singular random walks. More applications of this kernel method become available during the period of preparing revised versions of this paper, including [10], [9], [57], [58], and [62]. For exact tail asymptotics without a determination of the unknown generating function(s) or Laplace transformation function(s), different methods were used in the following studies: Abate and Whitt [1], Lieshout and Mandjes [41], Miyazawa and Rolski [52], Dai and Miyazawa [11].

Other methods for studying two-dimensional problems, including exact tail asymptotics, also exist, for example, based on large deviations, on properties of the Markov additive process (including matrix-analytic methods), or on asymptotic properties of the Green functions. References include Borovkov and Mogul'skii [5], McDonald [45], Foley and McDonald [20, 21, 22], Khanchi [27, 28], Adan et al. [2], Raschel [56], Miyazawa [48, 49, 50], Kobayashi and Miyazawa [29], Takahashi et al. [59], Haque [24], Miyazawa [47], Miyazawa and Zhao [53], Kroese et al. [33], Haque et al. [25], Li and Zhao [37], Motyer and Taylor [55], Li et al. [35], He et al. [26], Liu et al. [42], Tang and Zhao [60], Kobayashi et al. [31], among others. For more references, people may refer to a recent survey on tail asymptotics of multi-dimensional reflecting processes for queueing networks by Miyazawa [51].

The main focus of this paper is to propose a kernel method for exact tail asymptotics of random walks in the quarter plane following the ideal in [14], based on which a complete description of the exact tail asymptotics for stationary probabilities of a non-singular genus 1 random walk is obtained. We claim that the unknown generating function $\pi_1(x)$, or equivalently, $\pi_2(y)$, has either one or two dominant singularities, and a total of four types of exact tail asymptotics exists: (1) exact geometric decay; (2) a geometric decay multiplied by a factor of $n^{-1/2}$; (3) a geometric decay multiplied by a factor of $n^{-3/2}$; and (4) a geometric decay multiplied by a factor of n. These results are essentially not new (for examples see references [5, 20, 22, 49, 28]) except that the fourth type is missing from previous studies for the discrete random walk, but was reported for the continuous random walk in [11]. For the case of two dominant singularities with the same asymptotic property, a new periodic phenomena in the tail asymptotic property is discovered, which has not been reported in previous literature. For the tail asymptotic behaviour of the non-boundary joint probabilities along a coordinate direction, a new method based on recursive relationships of probability generating functions will be applied, which is an extension of the idea used in [40].

For an unknown generating function of probabilities, a Tauberian-like theorem is used as a bridge to link the asymptotic property of the function at its dominant singularities to the tail asymptotic property of its coefficient, or in our case, stationary probabilities. This theorem does not require the monitonicity in the probabilities, which is required by a standard Tauberian theorem and cannot be verified in general, or Heaviside operational calculus, which is usually very difficult to be rigorous. However, the price paid for applying the Tauberian-like theorem requires more in analyticity of the function and detailed information about all dominant singularities, or singularities on the circle of convergence. Therefore we need to provide information about how many singularities exist on the circle of convergence and their detailed properties, such as the nature of the singularity and the multiplicity in the case of the pole, for the random walk. It is not always true that only one singularity exists on the circle of convergence. Technical details are needed to address these issues.

The kernel method immediately leads to exact tail asymptotics in the boundary probabilities, in both directions, based on which exact tail asymptotics in a marginal distribution will become clear. However, it does not directly lead to exact tail asymptotic properties for the joint probabilities along a coordinate direction, except for the boundary probabilities as mentioned above. Therefore, further efforts are required. In this paper, we propose a method, based on difference equations of the unknown generating functions, to do the asymptotic analysis, which successfully overcomes the hurdle for exact tail asymptotics for joint probabilities.

The rest of the paper is organized into eight sections. In Section 2, after the model description, the so-called fundamental form for the random walk in the quarter plane is provided, together with a stability condition. Section 3 contains necessary properties for the two branches (or an algebraic function) defined by the kernel equation and for the branch points of the branches. These properties are either directly from [14] or its further refinements. Section 4 consists of six subsections for the purpose of characterizing the asymptotic properties of the unknown generating functions $\pi_1(x)$ and $\pi_2(y)$ at their dominant singularities. Specifically, two Tauberian-like theorems are introduced in subsection 1; the interlace between the two unknown generating functions is discussed in subsection 2, which plays a key role in the proposed kernel method; detailed properties for singularities of the unknown generating functions are obtained in subsections 3–5, which finally lead to the main theorem (Theorem 4.8) in this section provided in the last subsection. In Section 5, asymptotic analysis for the boundary generating functions is carried out, which directly leads to the tail asymptotics for the boundary probabilities in terms of the Tauberian-like theorem. In Section 6, based on the asymptotic results obtained for the generating function of boundary probabilities in the previous section, and the fundamental form, exact tail asymptotic properties for the two marginal distributions are provided. Exact tail asymptotic properties for joint probabilities along a coordinate direction is addressed in Section 7, which is not a direct result from the kernel method. Instead, we propose a difference equation method to carry out asymptotic analysis of a sequence of unknown generating functions. The last section contributes to concluding remarks and two examples by applying the kernel method.

2. Description of the Random Walk

The random walk in the quarter plane considered in this paper to demonstrate the kernel method is a reflected random walk or a Markov chain with the state space $\mathbb{Z}_+^2 = \{(m,n); m,n \text{ are non-negative integers}\}$. To describe this process, we divide the whole quadrant $\mathbb{Z}_+^2$ into four regions: the interior $S_+ = \{(m,n);\, m,n = 1,2,\ldots\}$, horizontal boundary $S_1 = \{(m,0);\, m = 1,2,\ldots\}$, vertical boundary $S_2 = \{(0,n);\, n = 1,2,\ldots\}$, and the origin $S_0 = \{(0,0)\}$, or

$\mathbb{Z}_+^2 = S_+ \cup S_1 \cup S_2 \cup S_0$. In each of these regions, the transition is homogeneous. Specifically, let X_+, X_1, X_2 and X_0 be random vectors having the distributions, respectively, $p_{i,j}$ with $i, j = 0, \pm 1$; $p_{i,j}^{(1)}$ with $i = 0, \pm 1$ and $j = 0,1$; $p_{i,j}^{(2)}$ with $i = 0,1$ and $j = 0, \pm 1$; and $p_{i,j}^{(0)}$ with $i, j = 0,1$. Then, the transition probabilities of the random walk (Markov chain) $L(t) = (L_1(t), L_2(t))$ are given by

$$P(L(t+1) = (m_2, n_2) \mid L(t) = (m_1, n_1)) = \begin{cases} P(X_+ = (m_2 - m_1, n_2 - n_1)), & \text{if}\,(m_2, n_2) \in S, (m_1, n_1) \in S_+, \\ P(X_k = (m_2 - m_1, n_2 - n_1)), & \text{if}\,(m_2, n_2) \in S, (m_1, n_1) \in S_k \ \text{with}\ k = 0,1,2. \end{cases}$$

2.1. Ergodicity conditions

A stability (ergodic) condition can be found in Theorem 3.3.1 of Fayolle et al. [14], which has been amended by Kobayashi and Miyazawa as Lemma 2.1 in [29]. This condition is stated in terms of the drift vectors defined by

$$M = (M_x, M_y) = (\sum_i i\,(\sum_j p_{i,j}), \sum_j j\,(\sum_i p_{i,j})),$$
$$M^{(1)} = (M_x^{(1)}, M_y^{(1)}) = (\sum_i i\,(\sum_j p_{i,j}^{(1)}), \sum_j j\,(\sum_i p_{i,j}^{(1)})),$$
$$M^{(2)} = (M_x^{(2)}, M_y^{(2)}) = (\sum_i i\,(\sum_j p_{i,j}^{(2)}), \sum_j j\,(\sum_i p_{i,j}^{(2)})).$$

Theorem 2.1. (Theorem 3.3.1 in [14] and Lemma 2.1 in [29]) *When $M \neq 0$, the random walk is ergodic if and only if one of the following three conditions holds:*

1. $M_x < 0$, $M_y < 0$, $M_x M_y^{(1)} - M_y M_x^{(1)} < 0$ *and* $M_y M_x^{(2)} - M_x M_y^{(2)} < 0$;
2. $M_x < 0$, $M_y \geq 0$, $M_y M_x^{(2)} - M_x M_y^{(2)} < 0$ *and* $M_x^{(1)} < 0$ if $M_y^{(1)} = 0$;
3. $M_x \geq 0$, $M_y < 0$, $M_x M_y^{(1)} - M_y M_x^{(1)} < 0$ *and* $M_y^{(2)} < 0$ if $M_x^{(2)} = 0$.

Readers may also refer to Theorem 6.1 in [7], to which the above theorem is essentially equivalent. Throughout the paper, we make the following assumption, unless otherwise specified:

Assumption 1. *The random walk $L(t)$ is irreducible, positive recurrent and aperiodic.*

Under Assumption 1, let $\pi_{m,n}$ be the unique stationary probability distribution of the random walk.

Remark 2.1. *It should be noted that for a stable random walk, the condition $M \neq 0$ is equivalent to that both sequences $\{\pi_{m,0}\}$ and $\{\pi_{0,n}\}$ are light-tailed (for example, see Lemma 3.3 of [29]), which is our focus of this paper. Therefore, Theorem 2.1 provides a necessary and sufficient stability condition for the light-tailed case.*

2.2. Fundamental form

Define the following generating functions of the probability sequences for the interior states, horizontal boundary states and vertical boundary states, respectively,

$$\pi(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \pi_{m,n} x^{m-1} y^{n-1},$$
$$\pi_1(x) = \sum_{m=1}^{\infty} \pi_{m,0} x^{m-1},$$
$$\pi_2(y) = \sum_{n=1}^{\infty} \pi_{0,n} y^{n-1}.$$

The so-called fundamental form of the random walk provides a functional equation relating the three unknown generating functions $\pi(x,y)$, $\pi_1(x)$ and $\pi_2(y)$. To state the fundamental form, we define

$$h(x,y) = xy\left(\sum_{i=-1}^{1}\sum_{j=-1}^{1} p_{i,j}x^i y^j - 1\right)$$
$$= a(x)y^2 + b(x)y + c(x) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y),$$

$$h_1(x,y) = x\left(\sum_{i=-1}^{1}\sum_{j=0}^{1} p_{i,j}^{(1)}x^i y^j - 1\right)$$

$$= a_1(x)y + b_1(x) = \tilde{a}_1(y)x^2 + \tilde{b}_1(y)x + \tilde{c}_1(y),$$

$$h_2(x,y) = y\left(\sum_{i=0}^{1}\sum_{j=-1}^{1} p_{i,j}^{(2)}x^i y^j - 1\right)$$

$$= \tilde{a}_2(y)x + \tilde{b}_2(y) = a_2(x)y^2 + b_2(x)y + c_2(x),$$

$$h_0(x,y) = \left(\sum_{i=0}^{1}\sum_{j=0}^{1} p_{i,j}^{(0)}x^i y^j - 1\right)$$

$$= a_0(x)y + b_0(x) = \tilde{a}_0(y)x + \tilde{b}_0(y),$$

where

$$a(x) = p_{-1,1} + p_{0,1}x + p_{1,1}x^2,$$

$$b(x) = p_{-1,0} - (1 - p_{0,0})x + p_{1,0}x^2,$$

$$c(x) = p_{-1,-1} + p_{0,-1}x + p_{1,-1}x^2,$$

$$\tilde{a}(y) = p_{1,-1} + p_{1,0}y + p_{1,1}y^2,$$

$$\tilde{b}(y) = p_{0,-1} - (1 - p_{0,0})y + p_{0,1}y^2,$$

$$\tilde{c}(y) = p_{-1,-1} + p_{-1,0}y + p_{-1,1}y^2,$$

$$a_1(x) = p_{-1,1}^{(1)} + p_{0,1}^{(1)}x + p_{1,1}^{(1)}x^2,\ b_1(x) = p_{-1,0}^{(1)} - \left(1 - p_{0,0}^{(1)}\right)x + p_{1,0}^{(1)}x^2,$$

$$\tilde{a}_1(y) = p_{1,0}^{(1)} + p_{1,1}^{(1)}y,\ \tilde{b}_1(y) = p_{0,0}^{(1)} - 1 + p_{0,1}^{(1)}y,\ \tilde{c}_1(y) = p_{-1,0}^{(1)} + p_{-1,1}^{(1)}y$$

$$a_2(x) = p_{0,1}^{(2)} + p_{1,1}^{(2)}x,\ b_2(x) = p_{0,0}^{(2)} - 1 + p_{1,0}^{(2)}x,\ c_2(x) = p_{0,-1}^{(2)} + p_{1,-1}^{(2)}x$$

$$\tilde{a}_2(y) = p_{1,-1}^{(2)} + p_{1,0}^{(2)}y + p_{1,1}^{(2)}y^2,\ \tilde{b}_2(y) = p_{0,-1}^{(2)} - \left(1 - p_{0,0}^{(2)}\right)y + p_{0,1}^{(2)}y^2,$$

$$a_0(x) = p_{0,1}^{(0)} + p_{1,1}^{(0)}x,\ b_0(x) = p_{1,0}^{(0)}x - \left(1 - p_{0,0}^{(0)}\right),$$

$$\tilde{a}_0(y) = p_{1,0}^{(0)} + p_{1,1}^{(0)}y,\ \tilde{b}_0(y) = p_{0,1}^{(0)}y - \left(1 - p_{0,0}^{(0)}\right).$$

The basic equation for the generating function of the joint distribution, or the fundamental form of the random walk, is given by

$$-h(x,y)\pi(x,y) = h_1(x,y)\pi_1(x) + h_2(x,y)\pi_2(y) + h_0(x,y)\pi_{0,0}. \tag{2.1}$$

The reason for the above functional equation to be called fundamental is largely due to the fact that through analysis of this equation, the unknown generating functions can be determined or expressed, for example, through algebraic methods and boundary value problems as illustrated in Fayolle et al. [14]. The kernel method presented here also starts with the fundamental form, but without expressing generating functions first.

Remark 2.2. *The generating function* $\pi(x,y)$ *is defined for the stationary probabilities* $\pi_{m,n}$ *with* $m,n>0$, *excluding the boundary probabilities. (2.1) was proved in (1.3.6) in* [14]. *Based on (2.1), one can also obtain a similar fundamental form using generating functions including boundary probabilities:* $\Pi(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\pi_{m,n}x^m y^n$, $\Pi_1(x)=\sum_{m=0}^{\infty}\pi_{m,0}x^m$ *and* $\Pi_2(y)=\sum_{n=0}^{\infty}\pi_{0,n}y^n$.

For the conclusion of this section, we can easily check the following expressions, some of which will be needed in later sections:

$$M_y=a(1)-c(1)=\tilde{a}'(1)+\tilde{b}'(1)+\tilde{c}'(1), M_x=\tilde{a}(1)-\tilde{c}(1)=a'(1)+b'(1)+c'(1), \tag{2.2}$$

$$M_y^{(1)}=a_1(1)=\tilde{a}_1'(1)+\tilde{b}_1'(1)+\tilde{c}_1'(1), M_x^{(1)}=\tilde{a}_1(1)-\tilde{c}_1(1)=a_1'(1)+b_1'(1), \tag{2.3}$$

$$M_y^{(2)}=a_2(1)-c_2(1)=\tilde{a}_2'(1)+\tilde{b}_2'(1), M_x^{(2)}=\tilde{a}_2(1)=a_2'(1)+b_2'(1)+c_2'(1). \tag{2.4}$$

3. Branch Points and Functions Defined by the Kernel Equation

The property of the random walk relies on the property of the kernel function h and functions h_1 and h_2. The kernel function plays a key role in the kernel method.

Definition 3.1. *A random walk is called non-singular if the kernel function* $h(x,y)$, *as a polynomial in the two variables* x *and* y *over real numbers, is irreducible (equivalently, if* $h=fg$ *then either* f *or* g *is a constant) and quadratic in both variables.*

Throughout the paper unless otherwise specified, we make the second assumption below.

Assumption 2. *The random walk considered is non-singular.*

The non-singular condition for a random walk is closely related to the irreducibility of the marginal processes $L_1(t)$ and $L_2(t)$, but they are not the same concept. A necessary and sufficient condition for a random walk to be singular is given, in terms of $p_{i,j}$, in Lemma 2.3.2 in [14]. Study on tail asymptotics for a singular random walk is either easier or similar to the non-singular case, which can be found in Li et al. [36].

The starting point of our analysis is the set of all pairs (x,y) satisfying the kernel equation, or

$$B=\{(x,y)\in\mathbb{C}^2 : h(x,y)=0\},$$

where $\mathbb{C}$ is the set of all complex numbers. The kernel function can be considered as a quadratic form in either x or y with the coefficients being functions of y or x, respectively. Therefore, the kernel equation can be written as

$$a(x)y^2+b(x)y+c(x)=\tilde{a}(y)x^2+\tilde{b}(y)x+\tilde{c}(y)=0. \tag{3.1}$$

For a fixed x, the two solutions to the kernel equation as a quadratic form in y are given by the quadratic formula if $a(x)\neq 0$. Denote $Y_0(x)$ to be the root with the smaller modulus, and $Y_1(x)$ the root with the greater modulus. It is clear that that $Y_0(x)=Y_1(x)$ if and only if the discriminant $D_1(x)=b^2(x)-4a(x)c(x)$ is zero. Also, notice that non-singularity implies that $a(x)\not\equiv 0$ and, therefore, only up to two values of x could lead to $a(x)=0$ since $a(x)$ is a polynomial of degree up to 2.

Similarly, for a fixed y, the two solutions to the kernel equation as a quadratic form in x are given by $X_0(y)$ and $X_1(y)$, and $X_0(y)=X_1(y)$ if and only if the discriminant $D_2(y)=\tilde{b}^2(y)-4\tilde{a}(y)\tilde{c}(y)$ is zero.

It is important to study the set B, or equivalently $Y(x)$ or $X(y)$, since for all $(x,y)\in B$ with $|\pi(x,y)|<\infty$, the right hand side of the fundamental form is also zero, which provides a relationship between the two unknown generating functions π_1 and π_2.

According to (ii) of Theorem 5.3.3 in [14], the functions $X_0(y)$ or $X_1(y)$ ($Y_0(x)$ and $Y_1(x)$) defined in this paper coincide the functions $X_0(y)$ or $X_1(y)$ (Y_0 and Y_1) in [14] due to the uniqueness of the continuity. They are the two branches of the algebraic function $X(y)$ ($Y(x)$) determined by the kernel equation.

The concept of branch points introduced below is important in the discussion of the property of $Y(x)$ ($X(y)$).

Definition 3.2. *A branch point of $Y(x)$ ($X(y)$) is a value of x (y) such that $D_1(x)=0$ ($D_2(y)=0$).*

To discuss the branch points, notice that the discriminant $D_1(x)$ ($D_2(y)$) is a polynomial of degree up to four. Since the two cases are symmetric, we discuss $D_1(x)$ in detail only. Rewrite $D_1(x)$ as

$$D_1(x)=d_4x^4+d_3x^3+d_2x^2+d_1x+d_0,$$

where

$$d_0=p_{-1,0}^2-4p_{-1,1}p_{-1,-1},$$

$$d_1=2p_{-1,0}(p_{0,0}-1)-4(p_{-1,1}p_{0,-1}+p_{0,1}p_{-1,-1}),$$

$$d_2=(p_{0,0}-1)^2+2p_{1,0}p_{-1,0}-4(p_{1,1}p_{-1,-1}+p_{1,-1}p_{-1,1}+p_{0,1}p_{0,-1}),$$

$$d_3=2p_{1,0}(p_{0,0}-1)-4(p_{1,1}p_{0,-1}+p_{0,1}p_{1,-1}),$$

$$d_4=p_{1,0}^2-4p_{1,1}p_{1,-1}.$$

It can be easily checked that $d_1\le 0$ and $d_3\le 0$.

When D_1 is a polynomial of degree 4 (or $d_4\ne 0$), there are four branch points, denoted by x_i (y_i), $i=1,2,3,4$. Without loss of generality, we assume that $|x_1|\le|x_2|\le|x_3|\le|x_4|$. When the degree of $D_1(x)$ is $d<4$, for convenience, we let $x_{d+k}=\infty$ for integer $k>0$ such that $d+k\le 4$. For example, if $d=3$, then $x_4=\infty$. This can be justified by the following: consider the polynomial $\tilde{D}_1(\tilde{x})=D_1(x)/x^4$ in $\tilde{x}$, where $\tilde{x}=1/x$. Then, $\tilde{x}=0$ is a d-tuple zero of $\tilde{D}_1(\tilde{x})$, and therefore $x=\infty$ can be viewed as a d-tuple zero of $D_1(x)$.

The following lemma characterizes the branch points of $Y(x)$ for all non-singular random walks, including the heave-tailed case, or the case of $M=0$.

Lemma 3.1. *1. For a non-singular random walk with $M_y\ne 0$, $Y(x)$ has two branch points x_1 and x_2 inside the unit circle and another two branch points x_3 and x_4 outside the unit circle. All these branch points lie on the real line. More specifically,*

(1) *if* $p_{1,0}>2\sqrt{p_{1,1}p_{1,-1}}$, *then* $1<x_3<x_4<\infty$;

(2) *if* $p_{1,0}=2\sqrt{p_{1,1}p_{1,-1}}$, *then* $1<x_3<x_4=\infty$;

(3) *if* $p_{1,0}<2\sqrt{p_{1,1}p_{1,-1}}$, *then* $1<x_3\le -x_4<\infty$, *where the equality holds if and only if* $d_1=d_3=0$.

Similarly,

(4) if $p_{-1,0} > 2\sqrt{p_{-1,1}p_{-1,-1}}$, *then* $0 < x_1 < x_2 < 1$;

(5) if $p_{-1,0} = 2\sqrt{p_{-1,1}p_{-1,-1}}$, *then* $x_1 = 0$ *and* $0 < x_2 < 1$;

(6) if $p_{-1,0} < 2\sqrt{p_{-1,1}p_{-1,-1}}$, *then* $0 < -x_1 \le x_2 < 1$, *where the equality holds if and only if* $d_1 = d_3 = 0$.

2. For a non-singular random walk with $M_y = 0$ *(in this case* $M_x \neq 0$ *since we are only considering the genus 1 case in this paper), either* $x_2 = 1$ *if* $M_x < 0$; *or* $x_3 = 1$ *if* $M_x > 0$. *In the latter case, the system is unstable.*

Proof. We only need to prove **3.** and **6.** since all other proofs can be found in Fayolle et al. [14] (Lemma 2.3.8 and Lemma 2.3.9). We provide details for **3.** since **6.** can be proved similarly. Suppose otherwise $x_3 > -x_4$. From $d_1 \le 0$ and $d_3 \le 0$, we obtain $D_1(-x_3) = -d_3x_3^3 - d_1x_3 > 0$. On the other hand, $D_1(-\infty) = -\infty$ since $d_4 < 0$, which implies that $D_1(x) = 0$ has a fifth root in $(-\infty, x_3)$, but this is impossible. The contradiction shows that $x_3 \le -x_4$. It is clear that the equality holds if and only if $d_1 = d_3 = 0$.

Remark 3.1. *Similar results hold for the branch points* y_i, $i = 1,2,3,4$, *of* $X(y)$.

Definition 3.3. $p_{i,j}$ *(*$p_{i,j}^{(k)}$*) is called X-shaped if* $p_{i,j} = 0$ *(*$p_{i,j}^{(k)} = 0$*) for all* i *and* j *such that* $|i + j| = 1$. *A random walk is called X-shaped if* $p_{i,j}$ *and also* $p_{i,j}^{(k)}$ *for* $k = 1,2$ *are all X-shaped.*

Remark 3.2. *X-shaped transitions imply a certain kind of symmetry. Specifically, this symmetry means that if* $x_{dom} > 0$ *is a dominant singularity for* $\pi_1(x)$ *then* $-x_{dom}$ *is also a dominant singularity. Depending on at either even or odd state, the combined "influence" of these two dominant singularities on the asymptotic property would be either doubled or disappeared. This periodic behaviour is characterized in (5.5)–(5.8). This is similar to a periodic Markov chain, for which the limiting behaviour should be considered at multiples of the period. In our case, if the limit is taken at* $2k$, *then* $\pi_{2k+1,0}$ *has the same behaviour as* $\pi_{n,0}$ *when the random walk is not X-shaped.*

Based on Lemma 3.1, we can prove the following result.

Corollary 3.1. $x_3 = -x_4$ *if and only if* $p_{i,j}$ *is X-shaped.*

Throughout the rest of the paper, we define $[x_3, x_4] = [-\infty, x_4] \cup [x_3, \infty]$ when $x_4 < -1$. Similarly, $[y_3, y_4] = [-\infty, y_4] \cup [y_3, \infty]]$ when $y_4 < -1$. We define the following cut planes:

$$\tilde{\mathbb{C}}_x = \mathbb{C}_x \setminus [x_3, x_4],$$
$$\tilde{\mathbb{C}}_y = \mathbb{C}_y \setminus [y_3, y_4],$$
$$\tilde{\tilde{\mathbb{C}}}_x = \mathbb{C}_x \setminus [x_3, x_4] \cup [x_1, x_2],$$
$$\tilde{\tilde{\mathbb{C}}}_y = \mathbb{C}_y \setminus [y_3, y_4] \cup [y_1, y_2],$$

where $\mathbb{C}_x$ and $\mathbb{C}_y$ are the complex planes for x and y, respectively.

A list of basic properties of Y_0 and Y_1 (X_0 and X_1) is provided in the following lemma.

Lemma 3.2. *1. For* $|x| = 1$, $|Y_0(x)| \le 1$ *and* $|Y_1(x)| \ge 1$, *with equality only possibly for* $x = \pm 1$. *For* $x = 1$, *we have*

$$Y_0(1) = \min\left(1, \frac{\sum p_{i,-1}}{\sum p_{i,1}} = \frac{c(1)}{a(1)}\right),$$

$$Y_1(1) = \max\left(1, \frac{\sum p_{i,-1}}{\sum p_{i,1}} = \frac{c(1)}{a(1)}\right);$$

for $x = -1$, *the equality holds only if* $p_{i,j}$ *is X-shaped, for which we have*

$$Y_0(-1) = -\min\left(1, \frac{\sum p_{i,-1}}{\sum p_{i,1}} = \frac{c(1)}{a(1)}\right),$$

$$Y_1(-1) = -\max\left(1, \frac{\sum p_{i,-1}}{\sum p_{i,1}} = \frac{c(1)}{a(1)}\right).$$

2. *The functions* $Y_i(x)$, $i = 0,1$, *are meromorphic in the cut plane* $\tilde{\tilde{\mathbb{C}}}_x$. *In addition,*

(a) $Y_0(x)$ *has two zeros and no poles. Hence* $Y_0(x)$ *is analytic in* $\tilde{\tilde{\mathbb{C}}}_x$;

(b) $Y_1(x)$ *has two poles and no zeros.*

(c) $|Y_0(x)| \le |Y_1(x)|$, *in the whole cut complex plane* $\tilde{\tilde{\mathbb{C}}}_x$, *and equality takes place only on the cuts.*

3. *The function* $Y_0(x)$ *can become infinite at a point* x *if and only if,*

(a) $p_{11} = p_{10} = 0$, *in this case,* $x = x_4 = \infty$; *or*

(b) $p_{-11} = p_{-10} = 0$, *in this case,* $x = x_1 = 0$.

Parallel conclusions can be made for functions $X_0(y)$ *and* $X_1(y)$.

All results in **1.** come from Lemma 2.3.4 and Lemma 5.3.1 in [14] except for the expressions for $Y_0(-1)$ and $Y_1(-1)$, which can be obtained in the same fashion as for $Y_0(1)$ and $Y_1(1)$; results in **2.** are given in (ii) of Theorem 5.3.3 in [14]; and the conclusion in **3.** is the same as in (iii) of Theorem 5.3.3 in [14].

Remark 3.3. *All the above properties can be directly obtained through elementary analysis of the square root function.*

Throughout the rest of the paper, unless otherwise specified, we make the following assumption:

Assumption 3. *All branch points* x_i *and* y_i, $i = 1,2,3,4$, *are distinct.*

A random walk satisfying Assumption 3 is called a genus 1 random walk.

Remark 3.4. *This assumption is equivalent to the assumption that the Riemann surface defined by the kernel equation has genus 1. The Riemann surface for the random walk is either genus 1 or genus 0. A necessary and sufficient condition for the random walk in the quarter plane to be genus 1 is given in Lemma 2.3.10 in [14]. Most of queueing application models are the case of genus 1. The genus 0 case can be analyzed similarly except for the heavy-tailed case, the case where* $M = 0$. *In general, analysis of the genus 0 case (except for the case of* $M = 0$*) could be less challenging since expressions for the unknown generating functions* $\pi_1(x)$ *and* $\pi_2(y)$ *are either explicit or less complex than for the genus 1 case, which can immediately lead to an analytic continuation of these unknown generating functions. Chapter 6 of [14] is devoted to the genus 0 case.*

Corollary 3.2. *For a non-singular genus 1 random walk, if* $p_{i,j}$ *is X-shaped, then all* $p_{1,1}$, $p_{1,-1}$, $p_{-1,1}$ *and* $p_{-1,-1}$ *are positive.*

If only one of $p_{1,1}$, $p_{1,-1}$, $p_{-1,1}$ and $p_{-1,-1}$ is zero, then the random walk is non-singular having genus 0 (Lemma 2.3.10 in [14]) and if at least two of them are zero, then the random walk is singular (Lemma 2.3.2 in [14]).

Corollary 3.3. *For a stable random walk with* $M \neq 0$,

1. *If* $p^{(1)}_{i,j}$ *is X-shaped, then* $p^{(1)}_{1,1}$ *and* $p^{(1)}_{-1,1}$ *cannot be both zero; and*

2. *If* $p^{(2)}_{i,j}$ *is X-shaped, then* $p^{(2)}_{1,1}$ *and* $p^{(2)}_{-1,-1}$ *cannot be both zero.*

along the upper edge of the slit $[x', x'']$ and then back to x' along the lower edge of the slit. In this way, we can define the following image contours:

$$\mathcal{L} = Y_0[\overrightarrow{\underleftarrow{x_1x_2}}], \quad \mathcal{L}_{ext} = Y_0[\overrightarrow{\underleftarrow{x_3x_4}}]; \tag{3.2}$$

$$\mathcal{M} = X_0[\overrightarrow{\underleftarrow{y_1y_2}}], \quad \mathcal{M}_{ext} = X_0[\overrightarrow{\underleftarrow{y_3y_4}}], \tag{3.3}$$

respectively. Furthermore, for an arbitrary simple closed curve $\mathcal{U}$, by $G_{\mathcal{U}}$ we denote the interior domain bounded by $\mathcal{U}$ and by $G^c_{\mathcal{U}}$ the exterior domain.

The properties of the above image contours provided in the following lemma are important for the interlace between the two unknown functions $\pi(x)$ and $\pi_2(y)$ discussed in the next section. To state the lemma, define the following determinant:

$$\Delta = \begin{vmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{00} & p_{0,-1} \\ p_{-1,1} & p_{-1,1} & p_{-1,1} \end{vmatrix}$$

Lemma 3.3. *For non-singular genus 1 random walk without branch points on the unit circle, we have the following properties:*

1. *The curve* $\mathcal{M}$ *and* $\mathcal{M}_{ext}$ *are simple, closed and symmetrical about the real axis in* $\mathbb{C}_x$ *plane. Moreover,*

(a) *If* $\Delta > 0$, *then*

$$[x_1, x_2] \subset G_{\mathcal{M}} \subset G_{\mathcal{M}_{ext}} \text{ and } [x_3, x_4] \subset G^c_{\mathcal{M}_{ext}};$$

(b) *If* $\Delta < 0$, *then*

$$[x_1, x_2] \subset G_{\mathcal{M}_{ext}} \subset G_{\mathcal{M}} \text{ and } [x_3, x_4] \subset G^c_{\mathcal{M}};$$

(c) *If* $\Delta = 0$, *then*

$$[x_1, x_2] \subset G_{\mathcal{M}_{ext}} = G_{\mathcal{M}} \text{ and } [x_3, x_4] \subset G^c_{\mathcal{M}}.$$

Entirely symmetric results hold for $\mathcal{L}$ *and* $\mathcal{L}_{ext}$.

2. *The branches* X_i *and* Y_i *have the following properties:*

(a) *Both* $X_0(y)$ *and* $Y_0(x)$ *are conformal mappings:* $G_{\mathcal{M}} - [x_1, x_2] \underset{X_0(y)}{\overset{Y_0(x)}{\rightleftarrows}} G_{\mathcal{L}} - [y_1, y_2]$;

(b) $X_0(y) \in G_{\mathcal{M}} \cup G_{\mathcal{M}_{ext}}$ *and* $X_1(y) \in G^c_{\mathcal{M}} \cup G^c_{\mathcal{M}_{ext}}$. *Symmetrically,* $Y_0(x) \in G_{\mathcal{L}} \cup G_{\mathcal{L}_{ext}}$ *and* $Y_1(x) \in G^c_{\mathcal{L}} \cup G^c_{\mathcal{L}_{ext}}$;

(c) *If* $G_{\mathcal{M}} \subset G_{\mathcal{M}_{ext}}$, *then*

$$X_0(Y_0(t)) = t, \text{ if } t \in G_{\mathcal{M}},$$

$$X_0(Y_0(t)) \neq t, \text{ if } t \in G^c_{\mathcal{M}} \text{ and } X_0(Y_0(G^c_{\mathcal{M}})) = G_{\mathcal{M}}.$$

Symmetrically, if $G_{\mathcal{L}} \subset G_{\mathcal{L}_{ext}}$, *then*

$$Y_0(X_0(t)) = t, \text{ if } t \in G_{\mathcal{L}},$$

$$Y_0(X_0(t)) \neq t, \text{ if } t \in G_{\mathcal{L}}^c \text{ and } Y_0(X_0(G_{\mathcal{L}}^c)) = G_{\mathcal{L}}.$$

Proof. A proof of the lemma can be found in Theorem 5.3.3 (i) and Corollary 5.3.5 in [14]. Parallel results when **1.** is a branch point (or both 1 and -1 are branch points) can be found in Lemma 2.3.6, Lemma 2.3.9 and Lemma 2.3.10 of [14].

Remark 3.5. *Results in this lemma can also be directly proved through elementary analysis without using advanced mathematical concepts used in [14].*

4. Asymptotic Analysis of the Two Unknown Functions $\pi_1(x)$ and $\pi_2(y)$

The key idea of the kernel method is to consider all $(x, y) \in B$ such that the right hand side of the fundamental form is also zero, which provides a relationship between the two unknown functions $\pi_1(x)$ and $\pi_2(y)$. Then, the interlace between the unknown functions $\pi_1(x)$ and $\pi_2(y)$ plays the key role in the asymptotic analysis of these two functions, from which exact tail asymptotics of the stationary distribution can be determined according to asymptotic analysis of the unknown function at its singularities and the Tauberien-like theorem.

4.1. Tauberian-like theorems

Various approaches, say probabilistic or non-probabilistic, including analytic or algebraic, are available for exact geometric decay. However, asymptotic analysis seems unavoidable for exact non-geometric decay. A Tauberian, or Tauberian-like, theorem provides a tool of connecting the asymptotic property at dominant singularities of an analytic function at zero and the tail property of the sequence of coefficients in the Taylor series of the function. In our case, an unknown generating function of a probability sequence is analytic at zero. Since these probabilities are unknown, in general, it cannot be verified that the probability sequence is (eventual) monotone, which is a required condition for applying a standard Tauberian theorem. The tool used in this paper is a Tauberian-like theorem, which does not require this monitonicity. Instead, it imposes some extra condition on analyticity of the unknown generating function.

Let $A(z)$ be analytic in $|z| < R$, where R is the radius of convergence of the function $A(z)$. We first consider a special case in which R is the only singularity on the circle of convergence.

Remark 4.1. *It should be noticed that for an analytic function at 0, if the coefficients of the Taylor expansion are all non-negative, then the radius* $R > 0$ *of convergence is a singularity of the function according to the well-known Pringsheim's Theorem.*

Definition 4.1. (Definition VI.1. in Flajolet and Sedgewick [16]) *For given numbers* $\varepsilon > 0$ *and* ϕ *with* $0 < \phi < \pi/2$, *the open domain* $\Delta(\phi, \varepsilon)$ *is defined by*

$$\Delta(\phi, \varepsilon) = \{ z \in \mathbb{C} : |z| < 1 + \varepsilon, z \neq 1, \arg|z - 1| > \phi \}. \tag{4.1}$$

A domain is a Δ*-domain at 1 if it is a* $\Delta(\phi, \varepsilon)$ *for some* $\varepsilon > 0$ *and* $0 < \phi < \pi/2$. *For a complex number* $\zeta \neq 0$, *a* Δ*-domain at* ζ *is defined as the image* $\zeta \cdot \Delta(\phi, \varepsilon)$ *of a* Δ*-domain* $\Delta(\phi, \varepsilon)$ *at 1 under the mapping* $z \mapsto \zeta z$. *A function is called* Δ*-analytic if it is analytic in some* Δ*-domain.*

Remark 4.2. *The region* $\Delta(\phi, \varepsilon)$ *is an intended disk with the radius of* $1 + \varepsilon$. *Readers may refer to Figure VI.6 in [16] for a picture of the region. Throughout the paper, without otherwise stated, the limit of a* Δ*-analytic function is always taken in the* Δ*-domain.*

Theorem 4.1. (Tauberian-like theorem for single singularity) *Let* $A(z) = \sum_{n \geq 0} a_n z^n$ *be analytic at 0 with* R *the radius of convergence. Suppose that* R *is a singularity of* $A(z)$ *on the circle of convergence such that* $A(z)$ *can be continued to a* Δ*-domain at* R. *If for a real number* $\alpha \notin \{0, -1, -2, \ldots\}$,

$$\lim_{z\to R}(1-z/R)^{\alpha} A(z)=g,$$

where g *is a non-zero constant, then,*

$$a_n \sim \frac{g}{\Gamma(\alpha)} n^{\alpha-1} R^{-n},$$

where $\Gamma(\alpha)$ *is the value of the gamma function at* α.

Proof. This is a immediate consequence of Corollary VI.1 in [16] after the transform $z \mapsto Rz$.

For the random walks studied in this paper, we will prove that the unknown generating function $\pi_1(x)$ ($\pi_2(y)$) has only one singularity on the circle of its convergence, except the X-shaped random walk for which the convergent radius R and $-R$ are the only singularities. To deal with the later case, we introduce the following Tauberian-like theorem for the case of multiple singularities.

Theorem 4.2. (Tauberian-like theorem for multiple singularities) *Let* $A(z)=\sum_{n\geq 0} a_n z^n$ *be analytic when* $|z|<R$ *and have a finite number of singularities* ζ_k, $k=1,2,\ldots,m$ *on the circle* $|z|=R$ *of convergence. Assume that there exists a* Δ*-domain* Δ_0 *at 1 such that* A *can be continued to intersection of the* Δ*-domains* ζ_k *at* ζ_k, $k=1,2,\ldots,m$*:*

$$D=\cap_{k=1}^{m}(\zeta_k\cdot\Delta_0).$$

If for each k, *there exists a real number* $\alpha_k \notin \{0,-1,-2,\ldots\}$ *such that*

$$\lim_{z\to\zeta_k}(1-z/\zeta_k)^{\alpha_k} A(z)=g_k,$$

where g_k *is a non-zero constant, then,*

$$a_n \sim \sum_{k=1}^{m} \frac{g_k}{\Gamma(\alpha_k)} n^{\alpha_k-1}\zeta_k^{-n}.$$

Proof. This is an immediate corollary of Theorem VI.5 in [16] for the case where α_k is real, $\beta_k=0$, $\sigma_k(z)=\tau_k(z)=(1-z)^{-\alpha_k}$ and $\sigma_{k,n}=\dfrac{g_k}{\Gamma(\alpha)} n^{\alpha_k-1}$.

4.2. Interlace of the two unknown functions $\pi_1(x)$ *and* $\pi_2(y)$

The interlace of the unknown functions $\pi_1(x)$ and $\pi_2(y)$ is a key for asymptotic analysis of these functions. Let

$$\Gamma_a=\{x\in\mathbb{C} \mid |x|=a\},$$

$$D_a=\{x: |x|<a\},$$

$$\overline{D}_a=\{x: |x|\leq a\}.$$

When $a=1$, we write $\Gamma=\Gamma_1$, $D=D_1$ and $\overline{D}=\overline{D}_1$.

We fist state two literature results on the continuation of the functions $\pi_1(x)$ and $\pi_2(y)$.

Lemma 4.1. (Theorem 3.2.3 in [14]) *For a stable non-singular random walk having genus 1,* $\pi_1(x)$ *is a meromorphic function in the complex cut plane* $\tilde{\mathbb{C}}_x$*. Similarly,* $\pi_2(y)$ *is a meromorphic function in the complex cut plane* $\tilde{\mathbb{C}}_y$.

This continuation result is crucial for tail asymptotic analysis. The following argument might be helpful to see why such a continuation exists. When the right hand side of the fundamental form is zero, the x and y are related, say through the function $Y_0(x)$. Therefore, x_3 is the dominant singularity if there are no other singularities exist in $(1, x_3)$. Based on the expression for $\pi_1(x)$ obtained from the fundamental form, all other singularities come from the zeros of $h_1(x, Y_0(x))$, which are poles of $\pi_1(x)$, or the singularities of $\pi_2(Y_0(x))$. A similar intuition holds for the function $\pi_2(y)$. Based on the above argument, it is reasonable to expect Lemma 4.1.

Remark 4.3. *An analytic continuation can be achieved through various methods. In [14] and [18], it was proved in terms of properties of Riemann surfaces. In [29] and [23], direct methods were used for a convergent region. For some cases, a simple proof exists by using the property of the conformal mapping* Y_0 *or* X_0. *For example, for the case of* $M_y > 0$ *and* $M_x < 0$, *we know, from Lemma 3.2-*

Lemma 4.2. (Lemma 2.2.1. in [14]) *Assume that the random walk is ergodic with* $M \neq 0$ *and the polynomial* $h(x, y)$ *is irreducible. Then, there exists an* $\varepsilon > 0$ *such that the functions* $\pi_1(x)$ *and* $\pi_2(y)$ *can be analytically continued up to the circle* $\Gamma_{1+\varepsilon}$ *in their respective complex plane. Moreover, they satisfy the following equation in* $D_{1+\varepsilon}^2 \cap B$ *:*

$$h_1(x, y)\pi_1(x) + h_2(x, y)\pi_2(y) + h_0(x, y)\pi_{0,0} = 0.$$

Proof. The analytic continuation is a direct consequence of Lemma 4.1 and the equation is directly from the fundamental form.

Theorem 4.3. *1. Function* $\pi_2(Y_0(x))$ *is meromorphic in the cut complex plane* $\tilde{\tilde{\mathbb{C}}}_x$. *Moreover, if* $Y_0(x_3)$ *is not a pole of* $\pi_2(y)$, *then* x_3 *is the dominant singularity of* $\pi_2(Y_0(x))$ *and there exist* $\varepsilon > 0$ *and* $0 < \phi < \pi/2$ *such that*

$$\lim_{x \to x_3} \pi_2(Y_0(x)) = \pi_2(Y_0(x_3)) \text{ and } \lim_{x \to x_3} \pi_2'(Y_0(x)) = \pi_2'(Y_0(x_3)),$$

where the limit is taken over the Δ*-domain at* x_3.

Similarly, $\pi_1(X_0(y))$ *is meromorphic in the cut complex plane* $\tilde{\tilde{\mathbb{C}}}_y$. *Moreover, if* $X_0(y_3)$ *is not a pole of* $\pi_1(x)$, *then* y_3 *is the dominant singularity of* $\pi_1(X_0(y))$ *and there exist* $\varepsilon > 0$ *and* $0 < \phi < \pi/2$ *such that*

$$\lim_{y \to y_3} \pi_1(X_0(y)) = \pi_1(X_0(y_3)) \text{ and } \lim_{y \to y_3} \pi_1'(X_0(y)) = \pi_1'(X_0(y_3)),$$

where the limit is taken over the Δ*-domain at* y_3.

2. In cut plane $\tilde{\tilde{\mathbb{C}}}_x$, *equation*

$$h_1(x, Y_0(x))\pi_1(x) + h_2(x, Y_0(x))\pi_2(Y_0(x)) + h_0(x, Y_0(x))\pi_{0,0} = 0, \tag{4.2}$$

holds except at a pole (if there is any) of $\pi_1(x)$ *or* $\pi_2(Y_0(x))$. *Therefore,*

$$\pi_1(x) = \frac{-h_2(x, Y_0(x))\pi_2(Y_0(x)) - h_0(x, Y_0(x))\pi_{0,0}}{h_1(x, Y_0(x))}, \tag{4.3}$$

except at zero of $h_1(x, Y_0(x))$, *or at a pole (if there is any) of* $\pi_1(x)$ *or* $\pi_2(Y_0(x))$.

Similarly, in the cut plane $\tilde{\tilde{\mathbb{C}}}_y$, *equation*

$$h_1(X_0(y), y)\pi_1(X_0(y) + h_2(X_0(y), y)\pi_2(y) + h_0(X_0(y), y)\pi_{0,0} = 0, \tag{4.4}$$

holds except at a pole (if there is any) of $\pi_2(y)$ *or* $\pi_1(X_0(y))$. *Therefore,*

$$\pi_2(y) = \frac{-h_1(X_0(y),y)\pi_1(X_0(y)) - h_0(X_0(y),y)\pi_{0,0}}{h_2(X_0(y),y)}, \tag{4.5}$$

except at a zero of $h_2(X_0(y),y)$, *or at a pole (if there is any) of* $\pi_2(y)$ *or* $\pi_1(X_0(y))$.

Proof. We only prove the result for functions of x and the result for functions of y can be proved in the same fashion.

1. From Lemma 3.1 and Lemma 4.1, $Y_0(x)$ is analytic in the cut complex plane $\tilde{\tilde{\mathbb{C}}}_x$ and $\pi_2(y)$ is meromorphic in the cut complex plane $\tilde{\mathbb{C}}_y$, which implies $\pi_2(Y_0(x))$ is meromorphic in $\tilde{\tilde{\mathbb{C}}}_x$ if $Y_0(x) \notin [y_3, y_4]$. According to Lemma 3.3-2(b), for all $x \in \mathbb{C}_x$, $Y_0(x) \in G_{\mathcal{L}} \cup G_{\mathcal{L}_{ext}}$ and according to Lemma 3.3-1, $[y_3, y_4] \subset (G_{\mathcal{L}} \cup G_{\mathcal{L}_{ext}})^c$, which confirms $Y_0(x) \notin [y_3, y_4]$. From the above, we have $\pi_2(y)$ is analytic at $Y_0(x_3)$, then the limits in **1.** are immediate results of the analytic properties of $\pi_2(Y_0(x))$.

2. Since both $\pi_1(x)$ and $\pi_2(Y_0(x))$ are meromorphic (proved in **1.**) and $Y_0(x)$ is analytic (Lemma 3.1) in $\tilde{\tilde{\mathbb{C}}}_x$, equation (4.2) holds in the cut plane $\tilde{\tilde{\mathbb{C}}}_x$ except at the poles of $\pi_1(x)$ or $\pi_2(Y_0(x))$.

Remark 4.4. *Let us extend the definition of* $\pi_1(x)$ *to* $x = x_3$ *by* $\pi_1(x_3) = \lim_{x\to x_3}\pi_1(x)$ *for* x *in the cut plane. We say that* x_3 *is a pole if the limit of* $\pi_1(x)$ *is infinite as* $x \to x_3$ *in the cut plane.*

According to the above interlacing property and the Tauberian-like theorem, for exact tail asymptotics of the boundary probabilities $\pi_{n,0}$ and $\pi_{0,n}$, we only need to carry out an asymptotic analysis at the dominant singularities of the functions $\pi_1(x)$ and $\pi_2(y)$, respectively. There are only two possible types of singularities, poles or branch points. We need to answer the following questions:

Q1. How many singularities on the circle of convergence (dominant singularities)?

Q2. What is the multiplicity of a pole?

Q3. Is the branch point also a pole?

For the random walk considered in this paper, we will answer all these questions. We will see that on the convergent circle, there is only one singularity or there are exactly two singularities. For the former, Theorem 4.1 will be applied, and for the latter, Theorem 4.2 will be applied.

4.3. Poles of $\pi_1(x)$

Parallel properties about poles of the function $\pi_2(y)$ can be obtained in the same fashion, which will not be detailed here.

Lemma 4.3. *1. Let* $x \in G_{\mathcal{M}} \cap (\overline{D})^c$, *then the possible poles of* $\pi_1(x)$ *in* $G_{\mathcal{M}} \cap (\overline{D})^c$ *are necessarily zeros of* $h_1(x, Y_0(x))$, *and* $|Y_0(x)| \le 1$.

2. Let $y \in G_{\mathcal{L}} \cap (\overline{D})^c$, *then the possible poles of* $\pi_2(y)$ *in* $G_{\mathcal{L}} \cap (\overline{D})^c$ *are necessarily zeros of* $h_2(X_0(x), y)$, *and* $|X_0(y)| \le 1$.

Proof: 1. When $x \in \mathcal{M}$, then $Y_0(x) = y \in [y_1, y_2]$. From Lemma 3.2, for $|x| = 1$, $|Y_0(x)| \le 1$. For $x \in G_{\mathcal{M}} \cap (\overline{D})^c$, it follows from the maximum modulus principle, we have $|Y_0(x)| \le 1$. Hence, $\pi_2(Y_0(x))$ is analytic in $G_{\mathcal{M}} \cap (\overline{D})^c$. From Theorem 4.3, if $h_1(x, Y_0(x)) \ne 0$, equation (4.3) holds, which implies that the possible poles of $\pi_1(x)$ in $G_{\mathcal{M}} \cap (\overline{D})^c$ are necessarily zeros of $h_1(x, Y_0(x))$.

2. The proof is similar.

Theorem 4.4. *Let* x_p *be a pole of* $\pi_1(x)$ *with the smallest modulus greater than one. Assume that* $|x_p| \le x_3$. *Then, one of the follow two cases must hold:*

1. x_p *is a zero of* $h_1(x, Y_0(x))$ *;*

2. $\tilde{y}_0 = Y_0(x_p)$ *is a zero of* $h_2(X_0(y), y)$ *and* $|\tilde{y}_0| > 1$.

Parallel results hold for a pole of $\pi_2(y)$.

Proof. Suppose that x_p is not a zero of $h_1(x, Y_0(x))$. According to equation (4.3) in Theorem 4.3, x_p must be a pole of $\pi_2(Y_0(x))$ and $|\tilde{y}_0| > 1$. Furthermore, by Lemma 4.3, $x_p \notin G_{\mathcal{M}}$. If $\tilde{y}_0$ is not a zero of $h_2(X_0(y), y)$, according to equation (4.5) in Theorem 4.3, $\tilde{y}_0$ must be a pole of $\pi_1(X_0(y))$, that is, $\tilde{x}_0 = X_0(\tilde{y}_0)$ is a pole of $\pi_1(x)$. It follows from Lemma 4.3 that $\tilde{x}_0 = X_0(\tilde{y}_0)$ is a zero of $h_1(x, Y_0(x))$ if $\tilde{x}_0 \in G_{\mathcal{M}}$. There are two possible cases: $\Delta > 0$ or $\Delta \le 0$. If $\Delta > 0$, by Lemma 3.3-1(a) and 2(c), $\tilde{x}_0 \in G_{\mathcal{M}}$. In the case of $\Delta \le 0$, according to Lemma 3.3-1(b), 1(c) and 2(b), we also have $\tilde{x}_0 \in G_{\mathcal{M}}$. However, this case is not possible, since otherwise according to Lemma 3.3-1 we would have $\tilde{x}_0 = x_p$ or $\tilde{x}_0 = -x_p$, both leading to a contradiction. This completes the proof.

Remark 4.5. *We will show in the next subsection that a pole of* $\pi_1(x)$ *with the smallest modulus in the disk* $|x| \le x_3$ *is real.*

4.4. Zeros of $h_1(x, Y_0(x))$

In this subsection, we provide properties of the zeros of the function $h_1(x, Y_0(x))$. The main result is stated in the following theorem.

Theorem 4.5. *For a non-singular random walk having genus 1, consider the following two possible cases:*

1. *Either* $p_{i,j}$ *or* $p_{i,j}^{(1)}$ *is not X-shaped. In this case, either* $h_1(x, Y_0(x))$ *has no zeros with modulus in* $(1, x_3]$, *or it has only one simple zero, say* x^*, *with modulus in* $(1, x_3]$, *and* x^* *is positive.*

2. *Both* $p_{i,j}$ *and* $p_{i,j}^{(1)}$ *are X-shaped. In this case, either* $h_1(x, Y_0(x))$ *has no zeros with modulus in* $(1, x_3]$, *or it has exact two simple zeros, namely,* $x^* > 0$ *(with modulus in* $(1, x_3]$*) and* $-x^*$, *both are zeros of* $h_1(x, Y_0(x))$ *or both are zeros of* $a(x)h_1(x, Y_1(x))$.

With this theorem and Theorem 4.4, we are able to apply the Tauberian-like theorem to characterize the tail asymptotic properties for the boundary probability sequence $\pi_{n,0}$. To show the above Theorem, we need the following several lemmas and two propositions. Instead of directly considering the function $f_0(x) = h_1(x, Y_0(x))$, we consider a polynomial $f(x)$, which is essentially the product of $f_0(x)$ and $f_1(x) = h_1(x, Y_1(x))$:

$$f(x) = f_0(x)\tilde{f}_1(x),$$

where $\tilde{f}_1(x) = a(x) f_1(x)$. It is easy to verify, by noticing

$$Y_0(x)Y_1(x) = \frac{c(x)}{a(x)} \text{ and } Y_0(x) + Y_1(x) = -\frac{b(x)}{a(x)},$$

that

$$f(x) = a(x)b_1^2(x) - b(x)b_1(x)a_1(x) + c(x)a_1^2(x) \tag{4.6}$$

$$= d_6x^6 + d_5x^5 + d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0. \tag{4.7}$$

Hence, a zero of $f_i(x)$, $i=0,1$, has to be a zero of $f(x)$, and any zero of $f(x)$ is either a zero of $f_0(x)$ or a zero of $\tilde{f}_1(x)=a(x)f_1(x)$.

We can also write

$$f(x)=a(x)[a_1(x)]^2R_-(x)R_+(x), \tag{4.8}$$

where

$$R_\pm(x)=F(x)\pm\frac{\sqrt{D_1(x)}}{2a(x)} \tag{4.9}$$

with

$$F(x)=\frac{b_1(x)}{a_1(x)}-\frac{b(x)}{2a(x)}. \tag{4.10}$$

Remark 4.6. *It can be easily seen that both $f_0(x)$ and $\tilde{f}_1(x)$ are analytic on the cut complex plan. In fact, the analyticity of $f_0(x)$ is obvious and the analyticity of $\tilde{f}_1(x)$ is due to the cancellation of the zeros of $a(x)$ and the pole of $f_1(x)$.*

All proofs for Lemmas 4.4–4.7 and for Proposition 4.1 and Proposition 4.2 are omitted here, which can be found in an earlier version of this paper archived at arXiv:1505.04425.

Lemma 4.4. *1.* (a) $Y_0'(1)=\frac{M_x}{-M_y}$ *if* $M_y<0$; (b) $Y_1'(1)=\frac{M_x}{-M_y}$ *if* $M_y>0$; *and* (c) $Y_1(1)=Y_0(1)$ *and* $x=1$ *is a branch point of* $Y_1(x)$ *and* $Y_0(x)$ *if* $M_y=0$. *In this case,* $Y_1'(1)$ *and* $Y_0'(1)$ *do not exist. Parallel results hold for functions* $X_k(y)$ *for* $k=0,1$.

2. If $M_y\neq 0$, then $f(x)$ has at least one non-unit zero in $[x_2,x_3]$ and 1 is a simple zero of $f(x)$. Parallel results holds for the case of $M_x\neq 0$.

Lemma 4.5. *1. Let z be a branch point of $Y_0(x)$. If $f(z)=0$, then z cannot be a repeated root of $f(x)=0$.*

2. $f(x)$ (therefore both $f_0(x)$ and $\tilde{f}_1(x)$) has (have) no zeros on the cuts, except possibly at a branch point. More specifically, $f(x)<0$ if $a(x)<0$ and $f(x)>0$ if $a(x)>0$.

3. $f_0(x)$ and $\tilde{f}_1(x)$ have no common zeros except possibly at a branch point or at zero.

4. Consider the random walk in Theorem 4.5-1. If $f_0(x)$ has a zero in $[-x_3,-1)$, then $f_0(x)$ has an additional (different) zero in $[-x_3,-1)$.

5. For the random walk in Theorem 4.5-1, if $|x|\in(1,x_3]$, then $|Y_0(-|x|)|<Y_0(|x|)$.

Lemma 4.6. *Consider the random walk in Theorem 4.5-1. If $M_y\le 0$, then $x=1$ is the only zero of $f_0(x)=h_1(x,Y_0(x))$ on the unit circle $|x|=1$. If $M_y>0$, then $f_0(x)$ has no zero on unit circle $|x|=1$.*

Remark 4.7. *From the proof of Lemma 4.6, we can see that for the random walk considered in Theorem 4.5-2, $f_0(x)$ has no zeros with non-zero imaginary part on the unit circle.*

The proof of Theorem 4.5 is based on detailed properties of the function $f(x)$ and also the powerful continuity argument to connect an arbitrary random walk to a simpler one. For using this continuity argument, we consider the following special random walk.

Special Random Walk. This is the random walk for which $p_{i,j}$ is cross-shaped (or $p_{i,j}=0$ whenever $|ij|=1$), and $p_{-1,1}^{(1)}=p_{-1,0}^{(1)}=0$. We first prove the counterpart result to Theorem 4.5 for the

Special Random Walk.

Proposition 4.1. *For the Special Random Walk, the following results hold:*

1. *$f(x)=0$ has six real roots with exact one non-unit root in $[x_2, x_3]$. More specifically, two roots are zero, two in $[x_2, x_3]$, one in $(-\infty, x_1]$, and one in $[x_4, \infty)$.*

2. *If $f_0(x)$ has a zero, say x^*, in $(1, \ x_3]$, then x^* is the only zero of $f_0(x)$ with modulus in $(1, \ x_3]$. Furthermore, $f_0(x)$ has no other zeros with modulus greater than 1 except possibly at $x = x_4$.*

For the random walk considered in Theorem 4.5-2, we first prove the following results.

Lemma 4.7. *For the random walk considered in Theorem 4.5-2 (or both $p_{i,j}$ and $p_{i,j}^{(1)}$ are X-shaped), $f(1) = f(-1) = 0$, and $f(x) = 0$ has two more real roots, say $0 < x_0 \neq 1$ and $-x_0$, and two complex roots.*

Proposition 4.2. *For the random walk considered in Theorem 4.5-2 (or both $p_{i,j}$ and $p_{i,j}^{(1)}$ are X-shaped), either the two complex zeros of $f(x)$ are zeros of $\tilde{f}_1(x) = a(x) f_1(x)$ or they are inside the unit circle.*

<u>Proof of Theorem 4.5</u>

1. For the random walk considered here (either $p_{i,j}$ or $p_{i,j}^{(1)}$ is not X-shaped), let

$$\mathbf{p} = (p_{-1,-1},\, p_{0,-1},\, p_{1,-1},\, p_{-1,0},\, p_{0,0},\, p_{0,1},\, p_{-1,1},\, p_{0,1},\, p_{1,1}),$$

$$\mathbf{p}^{(1)} = (p_{-1,0}^{(1)},\, p_{0,0}^{(1)},\, p_{0,1}^{(1)},\, p_{-1,1}^{(1)},\, p_{0,1}^{(1)},\, p_{1,1}^{(1)}).$$

Define

$$A = \{(\mathbf{p}, \mathbf{p}^{(1)}) : 0 \le p_{i,j}, p_{i,j}^{(1)} \le 1 \ \text{ and } \ \sum_{i,j} p_{i,j} = \sum_{i,j} p_{i,j}^{(1)} = 1\}.$$

For an arbitrary random walk for which either $p_{i,j}$ or $p_{i,j}^{(1)}$ is not X-shaped, let ρ be the corresponding point in A. We assume that $M_y \le 0$ for the random walk ρ (and a similar proof can be found for the case of $M_y > 0$). Let ρ_0 be an arbitrarily chosen point in A corresponding the Special Random Walk. We prove the result by contradiction. Suppose otherwise that the statement were not true. There would be three possible cases: (i) $Im(x^*) \neq 0$; (ii) $-x_3 \le x^* < -1$; and (iii) there exists $x_0 \in (1, x_3]$ with $x_0 \neq x^*$ such that $f_0(x_0) = 0$.

Case (i). Clearly, $\overline{x^*}$ is also a root of $f(x) = 0$. Choose a simple connected path ℓ in A to connect ρ to ρ_0 such that on ℓ (excluding ρ, but including ρ_0) $M_y < 0$. The zeros of $f(x)$ as a function of parameters in A are continues on ℓ. There are two possible cases: (a) the zero function $x_0(\theta)$ (with $x_0(\rho) = x^*$) never passes the unit circle when θ travels from ρ to ρ_0; and (b) $x_0(\theta)$ passes the unit circle at some point $\theta \in \ell$.

If (a) occurs, let θ_0 be the first point at which $x_0(\theta) = \overline{x}_0(\theta)$, where $\overline{x}_0(\theta)$ is the zero function with $\overline{x}_0(\rho) = \overline{x^*}$. If $\overline{x^*}$ is a zero of $\tilde{f}_1$, then f_0 and $\tilde{f}_1$ would have a common zero $x_0(\theta_0) = \overline{x}(\theta_0)$ at θ_0, which contradicts Lemma 4.5-3. Hence, the only possibility is that $\overline{x^*}$ is also a zero of f_0. From θ_0 on, both $x_0(\theta)$ and $\overline{x}(\theta)$ should always be zeros of f_0, since otherwise only at a branch point a zero of f_0 could be switched to a zero of $\tilde{f}_1$ and all branch points are real, which means that $x_0(\theta) = \overline{x}(\theta)$ is a branch point and a multiple roots, contradicting to Lemma 4.5-1. As θ_0 approaches ρ_0, it leads to a contradiction that two zeros of f_0 are in $(1, x_3]$.

If (b) occurs, we can assume that when $x_0(\theta)$ pases the unit circle it is a zero of f_0 based on the proof in (a). Then, f_0 has two zeros since 1 is always a zero of f_0 independent of the parameters (or θ) when $M_y < 0$, which is a different zero from $x_0(\theta)$. This contradicts to the fact that f_0 has only one zero at the unit circle.

Case (ii). In this case, $f_0(x)$ would have another zero in $[-x_3, -1)$ at ρ according to Lemma 4.5-4. Consider the same two cases (a) and (b) as in (i). We can then follow a similar proof to show that case (ii) is impossible.

Case (iii). A similar proof will show that the case is impossible.

2. This is a direct consequence of Lemma 4.7 and Proposition 4.2.

The following Lemma gives a necessary and sufficient condition under which $f_0(x) = h_1(x, Y_0(x))$ has a zero in $(1, x_3]$.

Lemma 4.8. *Assume $M_y \neq 0$. We have following results:*

1. If $f_0(x_3) \geq 0$, $f_0(x)$ has a zero in $(1, x_3]$;

2. If $f_0(x_3) < 0$, $f_0(x)$ has no zeros in $(1, x_3]$.

Proof. 1. There are two cases: $M_y > 0$ or $M_y < 0$. If $M_y > 0$, then $f_0(1) < 0$, which leads to the conclusion. If $M_y < 0$, then $f_0'(1) < 0$, which also leads to the conclusion since $f_0(1) = 0$ and $f_0(x_3) \geq 0$.

2. Again there are two cases: $M_y > 0$ or $M_y < 0$. By simple calculus, in either case, we obtain that if $f_0(x) = 0$ had a root in $(1, x_3]$, then it would have another root in $(1, x_3]$ since $f_0(x_3) < 0$. This contradicts to Theorem 4.5.

4.5. Zeros of $h_2(X_0(y), y)$

Following the same argument in the previous subsection, we have the following result:

Theorem 4.6. *For a non-singular random walk having genus 1, consider the following two possible cases:*

1. Either $p_{i,j}$ or $p_{i,j}^{(2)}$ is not X-shaped. In this case, either $h_2(X_0(y), y)$ has no zeros with modulus in $(1, y_3]$, or it has only one simple zero, say y^*, with modulus in $(1, y_3]$, and y^* is positive.

2. Both $p_{i,j}$ and $p_{i,j}^{(2)}$ are X-shaped. In this case, either $h_2(X_0(y), y)$ has no zeros with modulus in $(1, y_3]$, or it has exact two simple zeros, namely, $y^* > 0$ (with modulus in $(1, y_3]$) and $-y^*$, both are zeros of $g_0(y)$ or both are zeros of $g_1(y)$, where

$$g_0(y) = h_2(X_0(y), y) \qquad \text{and} \qquad g_1(y) = h_2(X_1(y), y).$$

From the above analysis, we know that if $h_1(x, Y_0(x))$ has a zero in $(1, x_3]$, then such a zero is unique. Similarly, if $h_2(X_0(y), y)$ has a zero in $(1, y_3]$, then such a zero is unique. For convenience, we make the following convention:

Convention 1. *Let x^* be the unique zero in $(1, x_3]$ of the function $h_1(x, Y_0(x))$, if such a zero exists, otherwise let $x^* = \infty$. Similarly Let y^* be the unique zero in $(1, y_3]$ of the function $h_2(X_0(y), y)$ if such a zero exists, otherwise let $y^* = \infty$.*

According to Theorem 4.4, the unique pole in $(1, x_3]$ of $\pi_1(x)$ is either x^*, or the image of the pole under Y_0 is a zero of $h_2(X_0(y), y)$. Our focus in this subsection is on this special case of y^*.

Theorem 4.7. *If the pole in $(1, x_3]$ of $\pi_1(x)$ is not x^*, then, it, denoted by $\tilde{x}_1$, satisfies:*

1. $\tilde{x}_1 = X_1(y^*)$, *where y^* is the unique zero in $(1, y_3]$ of the function $h_2(X_0(y), y)$;*

2. $\tilde{x}_1$ *is the only pole of $\pi_1(x)$ with modulus in $(1, x_3]$, except for the case where both $p_{i,j}$ and $p_{i,j}^{(2)}$ are X-shaped, for which $-\tilde{x}_1$ is the other pole of $\pi_1(x)$ with modulus in $(1, y_3]$.*

Proof. 1. Let $\tilde{x}$ be the solution of $y^* = Y_0(x)$. Then, $\tilde{x} = \tilde{x}_0 \overset{\Delta}{=} X_0(y^*)$ or $\tilde{x} = \tilde{x}_1 \overset{\Delta}{=} X_1(y^*)$. If $y^* \in G_{\mathcal{L}}$, then $\tilde{x} = \tilde{x}_0$ so that $y^* = Y_0(X_0(y^*))$. In this case, by Lemma 4.3, $\tilde{x}_0 < 1$. If $y^* \in G_{\mathcal{L}}^c$, then $\tilde{x} = \tilde{x}_1$ so that $y^* = Y_0(X_1(y^*)$ and $\tilde{x}_1 \in G_{\mathcal{M}}^c$.

2. It follows from the fact that the zero, y^*, of $h_2(X_0(y), y)$ in $(1, y_3]$ is unique and the fact that $y^* = Y_0(x)$ has only two possible solutions $\tilde{x}_0 < 1$ and $\tilde{x}_1$. In the case where both $p_{i,j}$ and $p_{i,j}^{(2)}$ are X-shaped, $-y^*$ is the other zero of $h_2(X_0(y), y)$ with either $-y^* = Y_0(-\tilde{x}_1)$ or $-y^* = Y_0(-\tilde{x}_0)$.

Corollary 4.1. *Let $\tilde{x}$ be a solution of $y^* = Y_0(x)$. In order for $\tilde{x}$ to be in $(1, x_3]$ we need $y^* \in G_{\mathcal{L}}^c$. Furthermore, we have $y^* < y_3$.*

Proof. The first conclusion is directly from the proof to Theorem 4.7 and the second one follows from that fact that by Lemma 3.3-1 and Lemma 3.3-2(b), there exists no $x \in (1, x_3]$ such that $y^* = y_3 = Y_0(x)$. Therefore, we should have $y^* < y_3$.

Convention2. *Let $\tilde{x}_1 = X_1(y^*)$ if the unique zero y^* in $(1, y_3]$ of the function $h_2(X_0(y), y)$ exists, otherwise let $\tilde{x}_1 = \infty$.*

4.6. Asymptotics behaviour of $\pi_1(x)$ and $\pi_2(y)$

In this subsection, we provide asymptotic behaviour of two unknown functions $\pi_1(x)$ and $\pi_2(y)$. We only provide details for $\pi_1(x)$, since the behaviour for $\pi_2(y)$ can be characterized in the same fashion.

It follows from the discussion so far that:

(1) If $p_{i,j}$ is not X-shaped, then, independent of the properties of $p_{i,j}^{(1)}$ and $p_{i,j}^{(2)}$, there is only one dominant singularity, which is the smallest one of x^*, $\tilde{x}_1$ and x_3. Here x^*, $\tilde{x}_1$ and x_3 are not necessarily all different.

(2) If $p_{i,j}$ is X-shaped, then both x_3 and $-x_3$ are branch points.

(a) If $p_{i,j}^{(1)}$ is not X-shaped, then $h_1(x, Y_0(x)$ has either no zero or one zero x^* in $(1, x_3]$; and if $p_{i,j}^{(1)}$ is X-shaped, then $h_1(x, Y_0(x)$ has either no zero or two zeros $x^* \in (1, x_3]$ and $-x^*$.

(b) Similar to (a), $h_2(X_0(y), y)$ has either no zero in $(1, y_3]$ or one zero y^* in it. For the latter, if $p_{i,j}^{(2)}$ is not X-shaped, then $\tilde{x}_1 = X_1(y^*)$ is the only pole of $\pi_2(Y_0(x))$ with modulus in $(1, x_3]$; and if $p_{i,j}^{(2)}$ is X-shaped, then $\tilde{x}_1 = X_1(y^*) \in (1, x_3]$ and $-\tilde{x}_1 = X_1(-y^*)$ are the only two poles of $\pi_2(Y_0(x))$ with modulus in $(1, x_3]$.

Therefore, in case (2), we either have only one dominant singularity or exactly two dominant singularities depending on which of x^*, $\tilde{x}_1$ and x_3 is smallest and the property of $p_{i,j}^{(k)}$, $k = 1, 2$.

The theorem in this subsection provides detailed asymptotic properties at a dominant singularity for all possible cases. For the purpose of presenting the main theorem, we first state a fact, which assures that no removable singularities exit in the proof to the main result (Theorem 4.8). This fact can be justified based on literature results for the decay rate (for example, [29, 50]). For the purpose of completeness, we provide a direct proof based on our analysis.

Lemma 4.9. *Recall that* $\tilde{y}_0 = Y_0(x^*)$. *Both* $h_2(x^*, Y_0(x^*))\pi_2(Y_0(x^*)) + h_0(x^*, Y_0(x^*))\pi_{0,0} \neq 0$ *and* $h_1(X_0(\tilde{y}_0), \tilde{y}_0)\ \pi(X_0(\tilde{y}_0)) + h_0(X_0(\tilde{y}_0), \tilde{y}_0)\pi_{0,0} \neq 0$.

Proof. We provide a proof to the first result and the second one can be proved similarly. Let

$$N(x) = h_2(x, Y_0(x))\pi_2(Y_0(x)) + h_0(x, Y_0(x))\pi_{0,0}.$$

We show $N(x^*) \neq 0$. Without loss of generality, we assume $M_y < 0$ (since we cannot have both $M_x \geq 0$ and $M_y \geq 0$). We divide the proof into two cases according to whether or not $h_2(X_0(y), y)$ has a zero in $(1, y_3]$. Once again, proofs to the two cases are similar, we only show $N(x^*) \neq 0$ when $h_2(X_0(y), y)$ has a zero, denoted by $\tilde{y}_1$, in $(1, y_3]$.

1. We first assume $M_x < 0$. Under this condition, we can show that $h_2(X_1(y), y) > 0$ for $y \in [y_2, y_3]$, and $h_2(X_0(y), y) > 0$, < 0 and > 0 when $y \in [y_2, 1)$, $(1, \tilde{y}_1)$ and $(\tilde{y}_1, y_3]$, respectively.

If $Y_0(x^*) > 1$, then $h_0(x^*, Y_0(x^*)) \geq 0$ since both $x^* > 1$ and $Y_0(x^*) > 1$. It follows from $x = X_1(Y_0(x))$ for $x \in (1, x_3]$ that

$$x^* = X_1(Y_0(x^*)) \neq X_0(Y_0(x^*))$$

and then

$$h_2(x^*, Y_0(x^*)) = h_2(X_1(Y_0(x^*)), Y_0(x^*)) > 0,$$

which implies that $N(x^*) \neq 0$.

If $Y_0(x^*) \leq 1$, then $x^* = X_0(Y_0(x^*))$. Hence, $h_2(x^*, Y_0(x^*)) = h_2(X_0(Y_0(x^*)), Y_0(x^*)) > 0$. However, in this case, we do not always has $h_0(x^*, Y_0(x^*)) \geq 0$. When $h_0(x^*, Y_0(x^*)) < 0$, we prove $N(x^*) \neq 0$ by contradiction. For this purpose, let us consider

$$r(x) = a(x)h_0(x, Y_0(x))h_0(x, Y_1(x)).$$

We can show that $r(x)$ has at most four zeros including $x = 1$. Let x_{h_0} be its smallest zero in $(x^*, \min(\tilde{x}_1, x_3))$. Suppose otherwise $N(x^*) = 0$, then x^* is a removable pole, which implies that $\pi_1(x)$ is analytic in $(x^*, \min(\tilde{x}_1, x_3))$. Hence,

$$\pi_1(x_{h_0}) = \frac{h_2(x_{h_0}, Y_0(x_{h_0}))\pi_2(Y_0(x_{h_0}))}{-h_1(x_{h_0}, Y_0(x_{h_0}))} > 0.$$

Since $h_0(x_{h_0}, Y_0(x_{h_0})) = 0$ and $x_{h_0} > 1$, $Y_0(x_{h_0}) < 1$. Therefore, $h_2(x_{h_0}, Y_0(x_{h_0})) > 0$. On the other hand, we also have $h_1(x_{h_0}, Y_0(x_{h_0})) > 0$ since $x^* < x_{h_0} < x_3$. This would yield $\pi_1(x_{h_0}) < 0$, which is a contradiction.

2. We then assume $M_x \geq 0$. In this case, $h_2(X_1(y), y) > 0$ for $y \in (1, y_3)$. If $Y_0(x^*) \geq 1$, in the same fashion as the above, we can show $N(x^*) \neq 0$. If $Y_0(x^*) < 1$, then it follows from $Y_0(x^*) \in [y_2, 1)$ that $h_2(x^*, Y_0(x^*)) = h_2(X_0(Y_0(x^*)), Y_0(x^*)) < 0$, which implies $N(x^*) \neq 0$.

Remark 4.8. *When we deal with a specific model, the non-zero conditions are usually much easier to verify since the number of possible cases is significantly reduced.*

Let x_{dom} be a dominant singularity of $\pi_1(x)$. Clearly, $|x_{dom}|=x^*$, $|x_{dom}|=\tilde{x}_1$ or $|x_{dom}|=x_3$. The following Theorem shows the behaviour of $\pi_1(x)$ at x_{dom}. Recall again that $\tilde{y}_0=Y_0(x^*)$.

Theorem 4.8. *For the function $\pi_1(x)$, a total of four types of asymptotics exist as x approaches to a dominant singularity of $\pi_1(x)$, based on the detailed property of the dominant singularity.*

Case 1: *If $|x_{dom}|=x^*<\min\{\tilde{x}_1,x_3\}$, or $|x_{dom}|=\tilde{x}_1<\min\{x^*,x_3\}$, or $|x_{dom}|=x^*=\tilde{x}_1=x_3$, then*

$$\lim_{x\to x_{dom}}\left(1-\frac{x}{x_{dom}}\right)\pi_1(x)=c_{0,1}(x_{dom}),$$

where $c_{0,1}(x_{dom})$ is a non-zero constant.

Case 2: *If $|x_{dom}|=x^*=x_3<\tilde{x}_1$ or $|x_{dom}|=\tilde{x}_1=x_3<x^*$, then*

$$\lim_{\frac{x}{x_{dom}}\to 1}\sqrt{1-x/x_{dom}}\,\pi_1(x)=c_{0,2}(x_{dom}),$$

where $c_{0,2}(x_{dom})$ is a non-zero constant.

Case 3: *If $|x_{dom}|=x_3<\min\{\tilde{x}_1,x^*\}$, then*

$$\lim_{x\to x_{dom}}\sqrt{1-x/x_{dom}}\,\pi_1'(x)=c_{0,3}(x_{dom}),$$

where $c_{0,3}(x_{dom})$ is a non-zero constant.

Case 4: *If $|x_{dom}|=x^*=\tilde{x}_1<x_3$, then*

$$\lim_{x\to x_{dom}}\left(1-\frac{x}{x_{dom}}\right)^2\pi_1(x)=c_{0,4}(x_{dom}),$$

where $c_{0,4}(x_{dom})$ is a non-zero constant.

Remark 4.9. *All the non-zero constant $c_{0,k}$ for $k=1,2,3,4$ can be explicitly expressed. We omit the detailed expressions here and also the proof to the theorem since they are routine and tedious. Readers may refer to [40] for similar proofs and to arXiv:1505.04425 for a proof of the theorem. We also emphasize here that according to Lemma 4.9, the imposed non-zero conditions to Theorem 4.8 in an earlier version of this paper (arXiv:1505.04425) can be removed and therefore.*

Remark 4.10. *It should be noted that the above theorem provides the asymptotic behaviour at a dominant singularity, either positive or negative.*

Remark 4.11. *The requirement of $Y_0(\tilde{x}_1)>1$ in the definition of $\tilde{x}_1$ is important. It is possible that there exists some $\hat{x}_1$, which satisfies $h_2(X_0(Y_0(\hat{x}_1)),Y_0(\hat{x}_1))=0$ but $Y_0(\hat{x}_1)\le 1$. In this case, $\hat{x}_1$ cannot be a singular point since $Y_0(x)$ can be continued to $\hat{x}_1$.*

5. Tail Asymptotics of Boundary Probabilities $\pi_{n,0}$ and $\pi_{0,n}$

Since $\pi_1(x)$ and $\pi_2(y)$ are symmetric, properties for $\pi_1(x)$ can be easily translated to the counterpart properties for $\pi_2(y)$. Therefore, tail asymptotics for the boundary probabilities $\pi_{0,n}$ can be directly obtained by symmetry.

The exact tail asymptotics of the boundary probabilities $\pi_{n,0}$ is a direct consequence of Theorem 4.8 and a Tauberian-like theorem applied to the function $\pi_1(x)$. Specifically, if $\pi_1(x)$ has only one dominant singularity, then Theorem 4.1 is applied; and if $\pi_1(x)$ has two dominant singularity, then

Theorem 4.2 is applied.

The following theorem shows that there are four types of exact tail asymptotics, for large n, together with a possible periodic property if $\pi_1(x)$ has two dominant singularities that have the same asymptotic property.

In the theorem, let x_{dom} be the positive dominant singularity of $\pi_1(x)$. Consider the following four cases regarding which of x^*, $\tilde{x}_1$ and x_3 will be x_{dom}:

Case 1: $x_{dom} = \min\{x^*, \tilde{x}_1\} < x_3$ with $x^* \neq \tilde{x}_1$, or $x_{dom} = \tilde{x}_1 = x^* = x_3$;

Case 2: $x_{dom} = x_3 = \min\{x^*, \tilde{x}_1\}$ with $x^* \neq \tilde{x}_1$;

Case 3: $x_3 = x_{dom} < \min\{x^*, \tilde{x}_1\}$;

Case 4: $x_{dom} = x^* = \tilde{x}_1 < x_3$.

Theorem 5.1. *Consider the stable non-singular genus 1 random walk. Corresponding to the above four cases, we have the following tail asymptotic properties for the boundary probabilities* $\pi_{n,0}$ *for large* n*. In all cases,* $c_{0,i}(x_{dom})$ ($1 \le i \le 4$) *are given in Theorem 4.8.*

1. *If* $p_{i,j}$ *is not X-shaped, then there are four types of exact tail asymptotics:*

Case 1: *(Exact geometric decay)*

$$\pi_{n,0} \sim c_{0,1}(x_{dom})\left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.1}$$

Case 2: *(Geometric decay multiplied by a factor of* $n^{-1/2}$*)*

$$\pi_{n,0} \sim \frac{c_{0,2}(x_{dom})}{\sqrt{\pi}} n^{-1/2}\left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.2}$$

Case 3: *(Geometric decay multiplied by a factor of* $n^{-3/2}$*)*

$$\pi_{n,0} \sim \frac{c_{0,3}(x_{dom})}{\sqrt{\pi}} n^{-3/2}\left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.3}$$

Case 4: *(Geometric decay multiplied by a factor of* n*)*

$$\pi_{n,0} \sim c_{0,4}(x_{dom}) n \left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.4}$$

2. *If* $p_{i,j}$ *is X-shaped, but both* $p_{i,j}^{(1)}$ *and* $p_{i,j}^{(2)}$ *are not X-shaped, we then have the following exact tail asymptotic properties:*

Case 1: *(Exact geometric decay) It is given by (5.1);*

Case 2: *(Geometric decay multiplied by a factor of* $n^{-1/2}$*) It is given by (5.2);*

Case 3: *(Geometric decay multiplied by a factor of* $n^{-3/2}$*)*

$$\pi_{n,0} \sim \frac{\left[c_{0,3}(x_{dom}) + (-1)^{n-1} c_{0,3}(-x_{dom})\right]}{\sqrt{\pi}} n^{-3/2}\left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.5}$$

Case 4: *(Geometric decay multiplied by a factor of* n*) It is given by (5.4);*

3. *If* $p_{i,j}$ *and* $p_{i,j}^{(1)}$ *are X-shaped, but* $p_{i,j}^{(2)}$ *is not, we then have the following exact tail asymptotic properties:*

Case 1: *(Exact geometric decay) When* $x^* \geq \tilde{x}_1$, *it is given by (5.1); when* $x_{dom} = x^* < \tilde{x}_1$, *it is given by*

$$\pi_{n,0} \sim \left[c_{0,1}(x_{dom}) + (-1)^{n-1} c_{0,1}(-x_{dom})\right]\left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.6}$$

Case 2: *(Geometric decay multiplied by a factor of* $n^{-1/2}$*) When* $x^* > \tilde{x}_1$, *it is given by (5.2); when* $x_{dom} = x^* < \tilde{x}_1$, *it is given by*

$$\pi_{n,0} \sim \frac{\left[c_{0,2}(x_{dom}) + (-1)^{n-1} c_{0,2}(-x_{dom})\right]}{\sqrt{\pi}} n^{-1/2}\left(\frac{1}{x_{dom}}\right)^{n-1}; \tag{5.7}$$

Case 3: *(Geometric decay multiplied by a factor of* $n^{-3/2}$*) It is given by (5.5).*

Case 4: *(Geometric decay multiplied by a factor of* n*) It is given by (5.4).*

4. *If* $p_{i,j}$ *and* $p_{i,j}^{(2)}$ *are X-shaped, but* $p_{i,j}^{(1)}$ *is not, then it is the symmetric case to **3**. All expression in **3** are valid after switching* x^* *and* $\tilde{x}_1$.

5. *If all* $p_{i,j}$, $p_{i,j}^{(1)}$ *and* $p_{i,j}^{(2)}$ *are X-shaped, we then have the following exact tail asymptotic properties:*

Case 1: *(Exact geometric decay) When* $x^* \leq \tilde{x}_1$, *it is given by (5.6); when* $x^* > \tilde{x}_1$, *it is also given by (5.6) by replacing the dominant singularity* x^* *by* $\tilde{x}_1$.

Case 2: *(Geometric decay multiplied by a factor of* $n^{-1/2}$*) When* $x^* < \tilde{x}_1$, *it is given by (5.7); when* $x^* > \tilde{x}_1$, *it is also given by (5.7) by replacing the dominant singularity* x^* *by* $\tilde{x}_1$.

Case 3: *(Geometric decay multiplied by a factor of* $n^{-3/2}$*) It is given by (5.5).*

Case 4: *(Geometric decay multiplied by a factor of* n*) It is given by*

$$\pi_{n,0} \sim \left[c_{0,4}(x_{dom}) + (-1)^{n-1} c_{0,4}(-x_{dom})\right] n \left(\frac{1}{x_{dom}}\right)^{n-1}. \tag{5.8}$$

Proof. 1. Since $p_{i,j}$ is not X-shaped, all $-x_3$, $-x^*$ and $-\tilde{x}_1$ are not dominant singularities according to Corollary 3.1, Theorem 4.5 and Theorem 4.6. Therefore, there is only one dominant singularity for $\pi_1(x)$. The tail asymptotic properties of $\pi_{n,0}$ follow from Theorem 4.8 and the direct application of the Tauberian-like theorem (Theorem 4.1).

2. We only provide a proof to the cases, which are not identical to that in 1.

Case 1. For the case that $x_{dom} = \tilde{x}_1 = x^* = x_3$, we notice that $-x_3$ is also a dominant singularity (Corollary 3.1). In this case, the Tauberian-like theorem (Theorem 4.2) is used to have a tail asymptotic expression consisting of two terms, one, corresponding to the positive dominant singularity, with the exact geometric decay rate and the other, corresponding to the negative dominant singularity, with the geometric decay rate multiplied by a factor of $n^{-3/2}$. Therefore, the term with the geometric decay rate is the dominant (decay slower) term leading to the same tail asymptotic property given in (5.1).

Case 2. Similar to Case 1, $-x_3$ is also a dominant singularity. The Tauberian-like theorem (Theorem 4.2) leads to a tail asymptotic expression consisting of two terms, one with the geometric rate multiplied by a factor of $n^{-1/2}$ (dominant term) and the other by $n^{-3/2}$.

Case 3. In this case, both x_3 and $-x_3$ are dominant singularities having the same asymptotic property according to Theorem 4.8. The tail asymptotic expression follows from the application of the Tauberian-like theorem (Theorem 4.2).

3. In this case, $-x_3$ and $-x^*$ are singularities, but $-\tilde{x}_1$ is not. We only provide a proof to the cases, which are not identical to that in 1 or in 2.

Case 1. For the case when $x^* = \tilde{x}_1 = x_3$, there are two dominant singularities. The Tauberian-like theorem (Theorem 4.2) leads to a tail asymptotic expression consisting of two terms, one (corresponding to the positive singularity) with a geometric decay rate, and the other (corresponding to the negative singularity) with the same geometric decay rate multiplied by a factor of $n^{-1/2}$ that is dominated by the geometric decay.

When $x^* < \tilde{x}_1$, both x^* and $-x^*$ are dominant singularities with the same asymptotic property, which leads to the tail asymptotic expression by using Theorem 4.2.

Case 2. For case when $x_3 = x^*$, there are two dominant singularities having the same asymptotic property. The tail asymptotic expression follows from Theorem 4.2.

Case 4. In this case, there are two dominant singularities, but the contribution from the positive dominant singularity dominates that from the negative dominant singularity. The tail asymptotic expression follows from Theorem 4.2.

4. The symmetric case to 3.

5. In this case, all $-x^*$, $-x^*$ and $-x_3$ are singularities. We only provide a proof to the cases, which are not considered in the above.

Case 1. The only new situation here is the case when $x^* = x^* = x_3$. In this case, we have the same asymptotic property at both dominant singularities, which leads to (5.6).

Case 4. In this case, we have the same asymptotic property at both dominant singularities, which leads to (5.8).

From the above theorem, it is clear that if there is only one dominant singularity, then the boundary probabilities $\pi_{n,0}$ have the following four types of astymptotics: **1.** exact geometric; **2.** geometric multiplied by a factor of $n^{-1/2}$; **3.** geometric multiplied by a factor of $n^{-3/2}$; and **4.** geometric multiplied by a factor of n. If there are two dominant singularities, but with different asymptotic properties, $\pi_{n,0}$ also has one of the above four types of tail asymptotic properties. Finally, if we have the same asymptotic property at both dominant singularities, then $\pi_{n,0}$ reveals a periodic property with the above four types of tail asymptotics, which is a new discovery.

6. Tail Asymptotics of the Marginal Distributions

In the previous section, we have seen that the asymptotic behaviour of the function $\pi_1(x)$ ($\pi_2(y)$) at its dominant singularity or singularities determines the tail asymptotic property of the boundary probabilities $\pi_{n,0}$ ($\pi_{0,n}$). According the the fundamental form of the random walk, it, together with the property of the kernel function $h(x,y)$, also determines the tail asymptotic property of the marginal distribution $\pi_n^{(1)} = \sum_j \pi_{n,j}$ (and $\pi_n^{(2)} = \sum_i \pi_{i,n}$).

In this section, we provide properties for the exact tail asymptotics of the marginal distribution $\pi_n^{(1)}$. The exact tail asymptotics of $\pi_n^{(2)}$ can be easily obtained by symmetry. First, based on the fundamental form, we have

$$\pi(x,y) = \frac{h_1(x,y)\pi_1(x) + h_2(x,y)\pi_2(y) + h_0(x,y)\pi_{0,0}}{-h(x,y)}$$

and therefore,

$$\pi(x,1) = \frac{h_1(x,1)\pi_1(x) + h_2(x,1)\pi_2(1) + h_0(x,1)\pi_{0,0}}{-h(x,1)}$$
$$= \frac{h_1(x,1)\pi_1(x) + h_2(x,1)\pi_2(1) + h_0(x,1)\pi_{0,0}}{-\tilde{a}(1)[x - X_0(1)][x - X_1(1)]}.$$

If $M_x \geq 0$, then $X_1(1) = 1$, which implies that the denominator of the expression for $\pi(x,1)$ does not have any zero outside the unit circle. In this case, $\pi_n^{(1)}$ has the same tail asymptotics as $\pi_{n,0}$. The only difference is the expression for the coefficient, which can be obtained from straight forward calculations.

If $M_x < 0$, then $X_0(1) = 1$ and $X_1(1) > 1$. If $p_{i,j}$ is not X-shaped, the analysis is so-called standard. If $p_{i,j}$ is X-shaped, then there are four subcases based on if $p_{i,j}^{(k)}$ is X-shaped or not. For these cases, detailed analysis varies, but similar. Instead of providing detailed analysis here, which is similar to that in previous sections and can be found in arXiv:1505.04425, we provide a summary of tail asymptotic properties for the marginal distribution $\pi_n^{(1)}$ for all possible cases. For this purpose, let x_{dom} be the positive dominant singularity of $\pi(x,1)$. Note that $X_1(1) \neq \tilde{x}_1$. The following are the all possible cases according to which of $\tilde{x}_1$, x^*, x_3 and $X_1(1)$ is x_{dom}.

Case A. $x_{dom} = \min\{\tilde{x}_1, x^*, x_3\} < X_1(1)$;

Case B. $x_{dom} = X_1(1) < \min\{\tilde{x}_1, x^*, x_3\}$;

Case C. $x_{dom} = X_1(1) = x^* < \min\{\tilde{x}_1, x_3\}$;

Case D. $x_{dom} = X_1(1) = x_3 < x^*$;

Case E. $x_{dom} = X_1(1) = x_3 = x^*$.

Remark 6.1. *The cases here are different from the cases classified in the previous section and the next section.*

The exact tail asymptotic properties are obtained according to the expression of $\pi(x,1)$ and the Taubarian-like theorem.

Theorem 6.1. *For the stable non-singular genus 1 random walk, the exact tail asymptotic properties for the marginal distribution* $\pi_n^{(1)}$, *as* n *is large, are summarized as:*

Case A: *This case includes Cases 1–4 in the previous section.* $\pi_n^{(1)}$ *has the same types of asymptotic properties as* $\pi_{n,0}$ *given in Theorem 5.1, respectively, with possible different expressions for the coefficients.*

Case B: $\pi_n^{(1)}$ *has an exact geometric decay.*

Case C: $\pi_n^{(1)}$ *has an exact geometric decay if* $Y_0(x^*) = 1$ *and a geometric decay multiplied by a factor of* n *if* $Y_1(x^*) = 1$, *respectively.*

Case D: $\pi_n^{(1)}$ *has an exact geometric decay.*

Case E: $\pi_n^{(1)}$ *has an exact geometric decay.*

7. Tail Asymptotics for Joint Probabilities

In the previous sections, we have seen how we can derive exact tail asymptotic properties for the boundary probabilities and for the marginal distributions based on the asymptotic property of $\pi_1(x)$

($\pi_2(y)$) and the kernel function. However, the exact tail asymptotic behaviour for joint probabilities cannot be obtained directly from them. Further tools are needed for this purpose. Our goal here is to characterize the exact tail asymptotics for $\pi_{n,j}$ for each fixed j and $\pi_{i,n}$ for each fixed i. Due to the symmetry, in this section, we provide details only for the former.

The relevant balance equations of the random walk are given by

$$(1-p_{0,0}^{(0)})\pi_{0,0} = p_{-1,0}^{(1)}\pi_{1,0} + p_{0,-1}^{(2)}\pi_{0,1} + p_{-1,-1}\pi_{1,1},$$

$$(1-p_{0,0}^{(1)})\pi_{1,0} = p_{1,0}^{(0)}\pi_{0,0} + p_{-1,0}^{(1)}\pi_{2,0} + p_{-1,-1}\pi_{2,1} + p_{1,-1}^{(2)}\pi_{0,1} + p_{0,-1}\pi_{1,1},$$

$$(1-p_{0,0}^{(1)})\pi_{i,0} = p_{1,0}^{(1)}\pi_{i-1,0} + p_{-1,0}^{(1)}\pi_{i+1,0} + p_{-1,-1}\pi_{i+1,1} + p_{1,-1}\pi_{i-1,1} + p_{0,-1}\pi_{i,1},\ \ i \ge 2,$$

$$(1-p_{0,0})\pi_{i,j} = p_{1,-1}\pi_{i-1,j+1} + p_{-1,-1}\pi_{i+1,j+1} + p_{0,-1}\pi_{i,j+1} + p_{1,0}\pi_{i-1,j} + p_{-1,0}\pi_{i+1,j}$$
$$+ p_{1,1}\pi_{i-1,j-1} + p_{0,1}\pi_{i,j-1} + p_{-1,1}\pi_{i+1,j-1},\ \ j \ge 2.$$

Let

$$\varphi_j(x) = \sum_{i=1}^{\infty}\pi_{i,j}x^{i-1},\quad j \ge 0, \qquad \psi_i(y) = \sum_{j=1}^{\infty}\pi_{i,j}y^{j-1},\quad i \ge 0.$$

From the above definition, it is clear that $\varphi_0(x) = \pi_1(x)$ and $\psi_0(y) = \pi_2(y)$. From the relevant balance equations, we obtain

$$c(x)\varphi_1(x) + b_1(x)\varphi_0(x) = a_0^*(x), \tag{7.1}$$

$$c(x)\varphi_2(x) + b(x)\varphi_1(x) + a_1(x)\varphi_0(x) = a_1^*(x), \tag{7.2}$$

$$c(x)\varphi_{j+1}(x) + b(x)\varphi_j(x) + a(x)\varphi_{j-1}(x) = a_j^*(x),\quad j \ge 2, \tag{7.3}$$

or

$$\varphi_{j+1}(x) = \frac{-b(x)\varphi_j(x) - a(x)\varphi_{j-1}(x) + a_j^*(x)}{c(x)},\quad j \ge 0, \tag{7.4}$$

where

$$a_0^*(x) = -c_2(x)\pi_{0,1} - b_0(x)\pi_{0,0},$$

$$a_1^*(x) = -c_2(x)\pi_{0,2} - b_2(x)\pi_{0,1} - a_0(x)\pi_{0,0},$$

$$a_j^*(x) = -c_2(x)\pi_{0,j+1} - b_2(x)\pi_{0,j} - a_2(x)\pi_{0,j-1},\quad j \ge 2.$$

First, we can prove the fact that a zero of $c(x)$ is not a pole of $\varphi_j(x)$ for all $j \ge 0$ (details are omitted here, but can be found in arXiv:1505.04425). Therefore $\varphi_j(x)$ has the same singularities as $\varphi_0(x)$. Based on this result, results obtained in previous sections, and the following lemma, we can prove our main result.

Lemma 7.1. *If* $\min\{x^*, \tilde{x}_1\} > x_3$, *then*

$$\lim_{x \to x_{dom}} \sqrt{1 - \frac{x}{x_{dom}}}\,\varphi_j'(x) = c_{3,j}(x_{dom}),$$

where $c_{3,0}(x_{dom})$ *is given in Theorem 4.8 and*

$$c_{3,j+1}(x_{dom}) = [A_3(x_{dom}) + B_3(x_{dom})j]\left(\frac{1}{Y_1(x_{dom})}\right)^j,\quad j \ge 0, \tag{7.5}$$

with

$$A_3(x_{dom}) = -\frac{c_{3,0}(x_{dom})b_1(x_{dom})}{c(x_{dom})}, \tag{7.6}$$

$$B_3(x_{dom}) = \frac{-h_1(x_{dom}, Y_0(x_{dom}))c_{3,0}(x_{dom})}{c(x_{dom})}. \tag{7.7}$$

Once again, we omit the proof here, which can be found in arXiv:1505.04425. We are now ready to prove the main theorem of this section, in which

$$A_1(x_{dom}) = -B_1(x_{dom}) + \frac{-c_{1,0}(x_{dom})b_1(x_{dom})}{c(x_{dom})} \tag{7.8}$$

$$= \left(\frac{h_1(x_{dom}, Y_0(x_{dom}))}{a(x_{dom})[Y_1(x_{dom}) - Y_0(x_{dom})]Y_0(x_{dom})} - \frac{b_1(x_{dom})}{c(x_{dom})} \right) c_{1,0}(x_{dom}),$$

$$A_2(x_{dom}) = -\frac{c_{2,0}(x_{dom})b_1(x_{dom})}{c(x_{dom})}, \tag{7.9}$$

and $A_3(x_{dom})$ is given in (7.6),

$$A_4(x_{dom}) = -\frac{b_1(x_{dom})c_{0,4}(x_{dom})}{c(x_{dom})}, \tag{7.10}$$

$$B_1(x_{dom}) = \frac{-h_1(x_{dom}, Y_0(x_{dom}))c_{1,0}(x_{dom})}{a(x_{dom})[Y_1(x_{dom}) - Y_0(x_{dom})]Y_0(x_{dom})}, \tag{7.11}$$

$$B_2(x_{dom}) = \frac{-c_{2,0}(x_{dom})h_1(x_{dom}, Y_0(x_{dom}))}{aY_0(x_{dom})^2}, \tag{7.12}$$

and $B_3(x_{dom})$ is given in (7.7).

Theorem 7.1. *Consider the stable non-singular genus 1 random walk. Corresponding to the four case, we then have the following tail asymptotic properties for the joint probabilities* $\pi_{n,j}$ *for large* n *.*

1. *If* $p_{i,j}$ *is not X-shaped, then there are four types of exact tail asymptotics:*

Case 1: *(Exact geometric decay)*

$$\pi_{n,j} \sim \left[A_1(x_{dom}) \left(\frac{1}{Y_1(x_{dom})} \right)^{j-1} + B_1(x_{dom}) \left(\frac{1}{Y_0(x_{dom})} \right)^{j-1} \right] \left(\frac{1}{x_{dom}} \right)^{n-1}, \quad j \geq 1; \tag{7.13}$$

Case 2: *(Geometric decay multiplied by a factor of* $n^{-1/2}$ *)*

$$\pi_{n,j} \sim \frac{A_2(x_{dom}) + (j-1)B_2(x_{dom})]}{\sqrt{\pi}} \left(\frac{1}{Y_1(x_{dom})} \right)^{j-1} n^{-1/2} \left(\frac{1}{x_{dom}} \right)^{n-1}, \quad j \geq 1; \tag{7.14}$$

Case 3: *(Geometric decay multiplied by a factor of* $n^{-3/2}$ *)*

$$\pi_{n,j} \sim \frac{A_3(x_{dom}) + (j-1)B_3(x_{dom})]}{\sqrt{\pi}} \left(\frac{1}{Y_1(x_{dom})} \right)^{j-1} n^{-3/2} \left(\frac{1}{x_{dom}} \right)^{n-1}, \quad j \geq 1; \tag{7.15}$$

Case 4: *(Geometric decay multiplied by a factor of* n *)*

$$\pi_{n,j} \sim \left[A_4(x_{dom}) \left(\frac{1}{Y_1(x_{dom})} \right)^{j-1} \right] n \left(\frac{1}{x_{dom}} \right)^{n-1}, \quad j \geq 1. \tag{7.16}$$

2. *If* $p_{i,j}$ *is X-shaped, but both* $p_{i,j}^{(1)}$ *and* $p_{i,j}^{(2)}$ *are not X-shaped, we then have the following exact tail asymptotic properties:*

Case 1: *(Exact geometric decay) It is given by (7.13);*

Case 2: *(Geometric decay multiplied by a factor of* $n^{-1/2}$*) It is given by (7.14);*

Case 3: *(Geometric decay multiplied by a factor of* $n^{-3/2}$*) It is given by*

$$\pi_{n,j} \sim \frac{A_3(x_{dom}) + (-1)^{n+j} A_3(-x_{dom})]}{\sqrt{\pi}} \left(\frac{1}{Y_1(x_{dom})}\right)^{j-1} n^{-3/2} \left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1; \tag{7.17}$$

Case 4: *(Geometric decay multiplied by a factor of* n*) It is given by (7.16).*

3. *If* $p_{i,j}$ *and* $p_{i,j}^{(1)}$ *are X-shaped, but* $p_{i,j}^{(2)}$ *is not, we then have the following exact tail asymptotic properties:*

Case 1: *(Exact geometric decay) When* $\tilde{x}_1 < x^*$*, it is given by (7.13); when* $\tilde{x}_1 = x^* = x_3$*, it is also given by (7.13); when* $x^* < \tilde{x}_1$*, it is given by*

$$\pi_{n,j} \sim \left[A_1(x_{dom}) + (-1)^{n+j} A_1(-x_{dom}) \right] \left(\frac{1}{Y_1(x_{dom})}\right)^{j-1} \left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1; \tag{7.18}$$

Case 2: *(Geometric decay multiplied by a factor of* $n^{-1/2}$*) When* $x^* > \tilde{x}_1$*, it is given by (7.14); when* $x^* < \tilde{x}_1$*, it is given by*

$$\pi_{n,j} \sim \frac{A_2(x_{dom}) + (-1)^{n+j} A_2(-x_{dom})]}{\sqrt{\pi}} \left(\frac{1}{Y_1(x_{dom})}\right)^{j-1} n^{-1/2} \left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1; \tag{7.19}$$

Case 3: *(Geometric decay multiplied by a factor of* $n^{-3/2}$*) It is given by*

$$\pi_{n,j} \sim \frac{A_3(x_{dom}) + (-1)^{n+j} A_3(-x_{dom})]}{\sqrt{\pi}} \left(\frac{1}{Y_1(x_{dom})}\right)^{j-1} n^{-3/2} \left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1;$$

Case 4: *(Geometric decay multiplied by a factor of* n*) It is given by (7.16).*

4. *If* $p_{i,j}$ *and* $p_{i,j}^{(2)}$ *are X-shaped, but* $p_{i,j}^{(1)}$ *is not, then it is the symmetric case to* **3**. *All expression in* **3.** *are valid after switching* x^* *and* $\tilde{x}_1$.

5. *If all* $p_{i,j}$, $p_{i,j}^{(1)}$ *and* $p_{i,j}^{(2)}$ *are X-shaped, we then have the following exact tail asymptotic properties:*

Case 1: *(Exact geometric decay) When* $x^* \leq \tilde{x}_1$*, it is given by (7.18); when* $x^* > \tilde{x}_1$*, it is also given by (7.18) by replacing the dominant singularity* x^* *by* $\tilde{x}_1$*;*

Case 2: *(Geometric decay multiplied by a factor of* $n^{-1/2}$*) When* $x^* < \tilde{x}_1$*, it is given by (7.19); when* $x^* > \tilde{x}_1$*, it is also given by (7.19) by replacing the dominant singularity* x^* *by* $\tilde{x}_1$*;*

Case 3: *(Geometric decay multiplied by a factor of* $n^{-3/2}$*) It is given by (7.17);*

Case 4: *(Geometric decay multiplied by a factor of* n*) It is given by*

$$\pi_{n,j} \sim \left[A_4(x_{dom}) + (-1)^{n+j} A_4(-x_{dom}) \right] \left(\frac{1}{Y_1(x_{dom})}\right)^{j-1} n \left(\frac{1}{x_{dom}}\right)^{n-1}, \quad j \geq 1.$$

Proof. 1.

Case 1: It follows from Section 4.6 that $\lim_{x\to x_{dom}}\left(1-\frac{x}{x_{dom}}\right)\varphi_0(x)=c_{0,1}(x_{dom})$. By the induction and equations (7.1)–(7.3), $\lim_{x\to x_{dom}}\left(1-\frac{x}{x_{dom}}\right)\varphi_j(x)=c_{1,j}(x_{dom})$ with

$$c_{1,1}(x_{dom})c(x_{dom})+c_{1,0}(x_{dom})b_1(x_{dom})=0,$$

$$c_{1,2}(x_{dom})c(x_{dom})+c_{1,1}(x_{dom})b(x_{dom})+c_{1,0}(x_{dom})a_1(x_{dom})=0,$$

$$c_{1,j+1}(x_{dom})c(x_{dom})+c_{1,j}(x_{dom})b(x_{dom})+c_{1,j-1}(x_{dom})a(x_{dom})=0,\quad j\geq 2.$$

Since $c_{1,j}(x_{dom})$, $j\geq 0$, satisfies the second order recursive relation above, it takes the form of

$$c_{1,j+1}(x_{dom})=A_1(x_{dom})\left(\frac{1}{Y_1(x_{dom})}\right)^j+B_1(x_{dom})\left(\frac{1}{Y_0(x_{dom})}\right)^j,\quad j\geq 0.$$

To determine $A_1=A_1(x_{dom})$ and $B_1=B_1(x_{dom})$, we use the initial equations:

$$(A_1+B_1)c(x_{dom})+c_{1,0}(x_{dom})b_1(x_{dom})=0, \tag{7.20}$$

$$\left[A_1\left(\frac{1}{Y_1(x_{dom})}\right)+B_1\left(\frac{1}{Y_0(x_{dom})}\right)\right]c(x_{dom})+(A_1+B_1)b(x_{dom})+c_{1,0}(x_{dom})a_1(x_{dom})=0. \tag{7.21}$$

Multiplying both sides of (7.21) by $Y_0(x_{dom})$, adding the resulting one to (7.20), and taking intoaccount $a(x_{dom})Y_0^2(x_{dom})+b(x_{dom})Y_0(x_{dom})+c(x_{dom})=0$, $h_1(x_{dom},Y_0(x_{dom}))=a_1(x_{dom})Y_0(x_{dom})+b_1(x_{dom})$ and $c(x_{dom})=Y_0(x_{dom})Y_1(x_{dom})a(x_{dom})$ yield:

$$(A_1+B_1)c(x_{dom})+c_{1,0}(x_{dom})b_1(x_{dom})=0,$$

$$A_1\frac{Y_0(x_{dom})}{Y_1(x_{dom})}c(x_{dom})+B_1c(x_{dom})+(A_1+B_1)b(x_{dom})Y_0(x_{dom})+c_{1,0}(x_{dom})a_1(x_{dom})Y_0(x_{dom})=0,$$

which gives (7.11) and (7.8). So, $B_1(x_{dom})=0$ if $x_{dom}=x^*$ and $B_1(x_{dom})\neq 0$ if $x_{dom}=\tilde{x}_1$. By the Tauberian-like theorem, we obtain (7.13).

Case 2: Similar to that for 1-Case 1. From the proof, we have (7.12) and (7.9).

Case 3: Write

$$\varphi_j'(x)=\sum_{n=0}^{\infty}(n+1)\pi_{n+2,j}x^n=\sum_{n=0}^{\infty}(n+1)x_3^n\pi_{n+2,j}\left(\frac{x}{x_3}\right)^n.$$

According Lemma 7.1 and the Tauberian-like theorem, we have

$$(n+1)x_3^n\pi_{n+2,j}\sim\frac{c_{3,j}(x_3)}{\sqrt{\pi}}n^{-1/2},$$

which is equivalent to (7.15).

Case 4: The results can be proved in the same fashion as in Case 1 and Case 2.

The proofs of the other cases are omitted due to the similarity to **1.** and Theorem 5.1.

8. Examples and Concluding Remarks

In this paper, for a non-singular genus 1 random walk, we proposed a kernel method to study the exact tail asymptotic behaviour of the joint stationary probabilities along a coordinate direction, when the value of the other coordinate is fixed, and also the exact tail asymptotic behaviour for the two marginal distributions. This work serves for multifold purposes:

1. Proposing an alternative method for exact tail asymptotic properties of random walks in the quarter plane, and also random walks in the half plane. Several applications have become available (e.g., [40, 36, 10, 9, 57, 58, 62]), demonstrating the effectiveness of this method.

2. Discovering a new periodic behaviour in tail asymptotics, which has not been reported before.

3. Completing previous studies (using a different method) on exact tail asymptotic properties reported in [50] by:

(a) Addressing the case left unsolved (see Remark 4.8 in [50]);

(b) Providing the missing type (Case 4 in Theorem 5.1) for the tail asymptoticproperty.

It should be mentioned here that an early version (which contained all key results in the current version) of this paper was completed in 2011, and we also noticed that item 3 has been addressed in [30], an independent work from our studies (which completes in the same year, 2011). In addition, in the final version of their studies, the authors included our findings of the new periodic tail behaviour, reported in this paper, and claimed that a case is missing from our study that was added to their paper for the completion. However, this case does not exist for a stable random walk (see Corollary 3.3), and our study here is complete.

4. Extending the difference equation method for exact tail asymptotic properties for joint probabilities along a coordinate direction (in addition to the boundary probabilities). This result is not a direct result from the kernel method.

For exact tail asymptotics, a total of four different types exist. The key idea of this kernel method is simple and the use of the Tauberian-like theorem greatly simplifies the analysis, which, unlike in the situation when a standard Tauberian theorem is used, is also rigorous. Under the assumption that there is only one dominant singularity, this method provides a straightforward routine analysis for the exact tail asymptotic behaviour. However, without this assumption, the analysis is not simple, at least to our best effort, for telling how many dominant singularities and when a pole is simple. It is also challenging to characterize the exact tail asymptotic along a coordinate direction when the value of the other coordinate is not zero, since it is not a direct consequence of the kernel method.

This kernel method can also be used for characterizing the exact tail asymptotics for the non-singular genus 0 case and the singular random walks (see Li, Tavakoli and Zhao [36]). With the detailed analysis provided in this paper, we expect further research in applying this kernel method to more general models.

The complete characterization of the exact tail asymptotic behaviour provided in this paper does not necessarily imply that for any specific model, a characterization explicitly in terms of the system parameters exists. However, we are confident that for any specific model, if using a different method could lead to a such characterization, in terms of system parameters, then it can be done using the kernel method. Finally, we mention two examples, which have been analyzed by using the proposed kernel method.

Example 1. A generalized two-demand model was considered in Li and Zhao [40] using the same idea proposed in this paper. For this model, let λ and λ_k ($k=1,2$) be the Poisson arrival rate with two demands and the arrival rate of the two dedicated Poisson arrivals, respectively. Furthermore, let μ_k ($k=1,2$) be the exponential service rates of the two independent parallel servers. For a detailed description of the model, one may refer to [40]. For this model, the three regions, on which the joint probabilities along a coordinate direction, say queue 1, have an exact geometric decay, a geometric decay multiplied by a factor $n^{-1/2}$ and a geometric decay multiplied by a factor $n^{-3/2}$ are extremely simple, which are: (a) $\frac{\mu_1}{\lambda+\lambda_1} < \frac{\mu_2-\lambda_2}{\lambda}$; (b) $\frac{\mu_1}{\lambda+\lambda_1} = \frac{\mu_2-\lambda_2}{\lambda}$; and (c) $\frac{\mu_1}{\lambda+\lambda_1} > \frac{\mu_2-\lambda_2}{\lambda}$, respectively.

Example 2. Consider the simple random walk, or a random walk for which $p_{i,j}$ and both $p_{i,j}^{(k)}$

($k = 1,2$) are cross-shaped. We then can follow the general results obtained in this paper to have refined properties. For example, consider the case of $M_y > 0$ and $M_x < 0$ and assume that the system is stable. Then, along the x-direction, $\pi_{n,j}$ has three types exact asymptotics in the following respective regions:

1. Exact geometric:

$$\frac{x_3}{x_3 - 1}\left[\sqrt{\frac{p_{0,-1}}{p_{0,1}}} - 1\right] p_{0,1}^{(1)} + p_{1,0}^{(1)} x_3 > p_{-1,0}^{(1)};$$

2. Geometric with a factor $n^{-1/2}$:

$$\frac{x_3}{x_3 - 1}\left[\sqrt{\frac{p_{0,-1}}{p_{0,1}}} - 1\right] p_{0,1}^{(1)} + p_{1,0}^{(1)} x_3 = p_{-1,0}^{(1)};$$

3. Geometric with a factor $n^{-3/2}$:

$$\frac{x_3}{x_3 - 1}\left[\sqrt{\frac{p_{0,-1}}{p_{0,1}}} - 1\right] p_{0,1}^{(1)} + p_{1,0}^{(1)} x_3 < p_{-1,0}^{(1)}.$$

When $M_y < 0$ and $M_x < 0$, this example also reveals the fourth type of exact tail asymptotic property, or a geometric decay multiplied by the factor n along the x-coordinate direction in the region defined by the following conditions:

$$\frac{x_3}{x_3 - 1}\left[\sqrt{\frac{p_{0,-1}}{p_{0,1}}} - 1\right] p_{0,1}^{(1)} + p_{1,0}^{(1)} x_3 \geq p_{-1,0}^{(1)}, \tag{8.1}$$

$$\frac{y_3}{y_3 - 1}\left[\sqrt{\frac{p_{-1,0}}{p_{1,0}}} - 1\right] p_{1,0}^{(2)} + p_{0,1}^{(2)} y_3 \geq p_{0,-1}^{(2)}, \tag{8.2}$$

$$h_1(x^*, \tilde{y}_0) = 0, \tag{8.3}$$

$$\frac{p_{-1,0}}{p_{1,0}} < \frac{p_{-1,0}^{(1)}}{p_{1,0}^{(1)}}, \tag{8.4}$$

and

$$\frac{(x^* - 1) p_{0,-1}^{(2)} p_{1,0} + p_{1,0}^{(2)} p_{0,-1}}{(x^* - 1) p_{0,1}^{(2)} p_{1,0} + p_{1,0}^{(2)} p_{0,1}} = 1 + \frac{(x^* - 1)[p_{-1,0}^{(1)} - p_{1,0}^{(1)} x^*]}{p_{0,1}^{(1)} x^*}. \tag{8.5}$$

Here, $x^* \in (1, x_3]$ and $y^* \in (1, y_3]$ are the zero $h_1(x, Y_0(x))$ and $h_2(X_0(y), y)$, respectively, whose existence is guaranteed by Lemma 4.8 under conditions (8.1) and (8.2); $\tilde{y}_0 = Y_0(x^*)$ and in this case we have $\tilde{y}_0 = y^*$; and $\tilde{x}_0 = X_0(Y_0(x^*))$.

It is not very difficult to see this is not an empty region. The last thing which we need to check is the coefficient

$$c_{0,4}(x_{dom}) = \frac{h_2(x_{dom}, y^*)[h_1(\tilde{x}_0, y^*)\pi(\tilde{x}_0) + h_0(\tilde{x}_0, y^*)]\pi_{0,0}}{x^{*2} h_1'(x_{dom}, y^*) Y_0'(x_{dom}) h_2'(X_0(y^*), y^*)} \neq 0, \tag{8.6}$$

or

$$h_1(\tilde{x}_0, y^*)\pi(\tilde{x}_0) + h_0(\tilde{x}_0, y^*)\pi_{0,0} \neq 0,$$

which is true since $h_2(x_{dom}, y^*) = h_2(X_1(y^*), y^*) > h_2(X_0(y^*), y^*) = 0$.

Before we conclude this paper, it is believed that the following concluding remarks are valuable: There is no trivial method for exact tail asymptotics for a two-dimensional problem. It is also true even for the rough decay. To the best of our knowledge, among all other available methods, only the method by Miyazawa (see [50] and [30]) and the kernel method proposed in this paper are systematic studies of exact tail asymptotics for the two-dimensional random walks in the quarter plane. The method proposed by Miyazawa has a geometric interpretation in the analysis, while the kernel method is purely analytic. The former method separate the analysis into two steps: finding the rough decay rate is the first step, and obtaining the exact tail asymptotic properties is the next (which follows the same idea, using Tauberian-like theorems as in this paper). The kernel method combines the above two steps together. Therefore, the focus looks simpler, which can be very important when the method is extended for more generalized problems (say random walks in the half plane).

As remarked by the authors of the book [14] (see Chapter 7), the principle of the kernel method is also valid for more general models (including higher-dimensional random walks). However, much more efforts are needed for technical details for higher-dimensional cases. For example, in general the analytic continuation of the unknown generating functions is still unaddressed for $k-$dimensional cases, where $k>2$.

Acknowledgements

The authors thank the anonymous referees for the valuable comments and suggestions, which significantly improved the quality of the paper, and the late Dr. P. Flajolet of INRIA for the discussion of the Tauberian-like theorem. This work was supported in part by Discovery Grants from NSERC of Canada. We would also acknowledge that many constructive suggestions/comments were made by an anonymous referee when an early version of this paper was submitted to a different outlet in 2011. Readers may refer to an early version of this paper at arXiv:1505.04425.

References

[1] Abate, J., & Whitt, W. (1997). Asymptotics for M/G/1 low-priority waiting-time tail probabilities. *Queueing Systems*, 25, 173–233.

[2] Adan, I., Foley, R. D., & McDonald, D. R. (2009). Exact asymptotics for the stationary distribution of a Markov chain: a production model. *Queueing Systems*, 62, 311–344.

[3] Banderier, C., Bousquet-Mélou, M, Denise, A., Flajolet, P., Gardy, D., & Gouyou-Beauchamps, D. (2002). Generating functions of generating trees. *Discrete Mathematics*, 246, 29–55.

[4] Bender, E. (1974). Asymptotic methods in enumeration, *SIAM Review*, 16, 485–513.

[5] Borovkov, A. A., & Mogul'skii, A. A. (2001). Large deviations for Markov chains in the positive quadrant. *Russian Mathematical Surveys*, 56, 803–916.

[6] Bousquet-Mélou, M. (2005). Walks in the quarter plane: Kreweras algebraic model. *Annals of Applied Probability*, 15, 1451-1491.

[7] Cohen, J. W. (1992). Analysis of Random Walks. IOS Press, Amsterdam.

[8] Cohen, J. W., & Boxma, O. J. (1983). Boundary Value Problems in Queueing System Analysis. North-Holland, Amsterdam.

[9] Dai, H., Dawson, D. A., & Zhao, Y. Q. (2015). Kernel method for staionary tails: from discrete to continuous. in *Asymptotic Laws and Methods in Stochastics*, edited by D. Dawson, R. Kulik, M. Ould Haye, B. Szyszkowicz and Y. Zhao, 297–328, Springer, New York.

[10] Dai, H., & Zhao, Y. Q. (2013). Wireless 3-hop networks with stealing revisited: A kernel approach. *INFOR*, 51(4), 192-205.

[11] Dai, J., & Miyazawa, M. (2011). Reflecting Brownian motion in two dimensions: Exact asymptotics for the stationary distribution. *Stochastic systems*, 1, 146–208

[12] Fayolle, G., & Iasnogorodski, R. (1979) Two coupled processors: the reduction to a Riemann-Hilbert problem. *Z. Wahrscheinlichkeitsth*, 47, 325–351.

[13] Fayolle, G., King, P. J. B., & Mitrani, I. (1982). The solution of certain two-dimensional Markov models. *Advances in Applied Probability*, 14, 295–308.

[14] Fayolle, G., Iasnogorodski, R., & Malyshev, V. (1999). Random Walks in the Quarter-Plane. Springer, New York.

[15] Flajolet, P., & Odlyzko, A. (1990). Singularity analysis of generating functions. *SIAM Journal of Discrete Mathematics*, 3, 216–240.

[16] Flajolet, F., & Sedgewick, R. (2009). Analytic Combinatorics. Cambridge University Press.

[17] Flatto, L., & McKean, H. P. (1977). Two queues in parallel. *Communications on Pure and Applied Mathematics*, 30, 255–263.

[18] Flatto, L., & Hahn, S. (1984). Two parallel queues created by arrivals with two demands I. *SIAM Journal on Applied Mathematics*, 44, 1041–1053.

[19] Flatto, L. (1985). Two parallel queues created by arrivals with two demands II. *SIAM Journal on Applied Mathematics*, 45, 861–878.

[20] Foley, R. D., & McDonald, D. R. (2001). Join the shortest queue: stability and exact asymptotics. *Annals of Applied Probability*, 11, 569–607.

[21] Foley, R. D., & McDonald, R. D. (2005). Large deviations of a modified Jackson network: stability and rough asymptotics. *Annals of Applied Probability*, 15, 519–541.

[22] Foley, R. D., & McDonald, R. D. (2005). Bridges and networks: exact asymptotics. *Annals of Applied Probability*, 15, 542–586.

[23] Guillemin, F., & van Leeuwarden, J. S. H. (2011). Rare event asymptotics for a random walk in the quarter plane. *Queueing Systems*, 67(1), 1–32.

[24] Haque, L. (2003).Tail Behaviour for Stationary Distributions for Two-Dimensional. Stochastic Models, Ph.D. Thesis. Carleton University, Ottawa, ON, Canada.

[25] Haque, L., Liu, L., & Zhao, Y. Q. (2005). Sufficient conditions for a geometric tail in a QBD process with countably many levels and phases. *Stochastic Models*, 21(1), 77–99.

[26] He, Q., Li, H., & Zhao, Y. Q. (2009). Light-tailed behaviour in QBD process with countably many phases. *Stochastic Models*, 25, 50–75.

[27] Khanchi, A. (2008). State of a network when one node overloads, Ph.D. Thesis. University of Ottawa, ON, Canada.

[28] Khanchi, A. (2009). Asymptotic hitting distribution for a reflected random walk in the positive quardrant. *Stochastic Models*, 27, 169–201.

[29] Kobayashi, M., & Miyazawa, M. (2014). Tail asymptotics of the stationary distribution of a two dimensional reflecting random walk with unbounded upward jumps. *Advances in Applied Probabability*, 46(2), 365–399.

[30] Kobayashi, M., & Miyazawa, M. (2013). Revisiting the tail asymptotics of the double QBD process: Refinement and complete solutions for the coordinate and diagonal directions. in *Matrix-Analytic Methods in Stochastic Models*, edited by G. Latouche, V. Ramaswami, J. Sethuraman, K. Sigman, M.S. Squillante and D. Yao, 145–185, Springer.

[31] Kobayashi, M., Miyazawa, M., & Zhao, Y. Q. (2010). Tail asymptotics of the occupation measure for a Markov additive process with an M/G/1-type background process. *Stochastic Models*, 26, 463–486.

[32] Knuth, D. E. (1969). *The Art of Computer Programming, Fundamental Algorithms*, Vol. 1, second ed., Addison-Wesley.

[33] Kroese, D. P., Scheinhardt, W. R. W., & Taylor, P. G. (2004). Spectral properties of the tandem Jackson network, seen as a quasi-birth-and-death process. *Annals of Applied Probability*, 14(4), 2057–2089.

[34] Kurkova, I. A., & Suhov, Y. M. (2003). Malyshev's theory and JS-queues. Asymptotics of stationary probabilities. *Annals of Applied Probability*, 13, 1313-1354.

[35] Li, L., Miyazawa, M., & Zhao, Y. (2007). Geometric decay in a QBD process with countable background states with applications to a join-the-shortest-queue model. *Stochastic Models*, 23, 413–438.

[36] Li, H., Tavakoli, J., & Zhao, Y. Q. (2013). Analysis of exact tail asymptotics for singular random walks in the quarter plane. *Queueing Systems*, 74, 151–179.

[37] Li, H., & Zhao, Y. Q. (2005). A retrial queue with a constant retrial rate, server break downs and impatient customers. *Stochastic Models*, 21, 531–550.

[38] Li, H., & Zhao, Y. Q. (2009). Exact tail asymptotics in a priority queue-characterizations of the preemptive model. *Queueing Systems*, 63, 355–381.

[39] Li, H., & Zhao, Y. Q. (2011). Exact tail asymptotics in a priority queue-characterizations of the non-preemptive model. *Queueing Systems*, 68, 165–192.

[40] Li, H.,& Zhao, Y. Q. (2011). Tail asymptotics for a generalized two demand queueing model-A kernel method. *Queueing Systems*, 69, 77–100.

[41] Lieshout, P., & Mandjes, M. (2008). Asymptotic analysis of Lévy-driven tandem queues. *Queueing Systems*, 60, 203–226.

[42] Liu, L., Miyazawa, M., & Zhao, Y. Q. (2008). Geometric decay in level-expanding QBD models. *Annals of Operations Research*, 160, 83–98.

[43] Malyshev, V. A. (1972). An analytical method in the theory of two-dimensional positive random walks. *Siberian Math. Journal*, 13, 1314–1329.

[44] Malyshev, V. A. (1973). Asymptotic behaviour of stationary probabilities for two dimensional positive random walks. *Siberian Math. Journal*, 14, 156–169.

[45] McDonald, D. R. (1999). Asymptotics of first passage times for random walk in an orthant. *Annals of Applied Probability*, 9, 110–145.

[46] Mishna, M. (2009). Classifying lattice walks restricted to the quarter plane. *Journal of Combinatorial Theory, Series A*, 116, 460–477.

[47] Miyazawa, M. (2004). The Markov renewal approach to M/G/1 type queues with countably many

background states. *Queueing Systems*, 46, 177–196.

[48] Miyazawa, M. (2007). Doubly QBD process and a solution to the tail decay rate problem. *Proceedings of the Second Asia-Pacific Symposium on Queueing Theory and Network Applications*, Kobe, Japan.

[49] Miyazawa, M. (2009). Two sided DQBD process and solutions to the tail decay rate problem and their applications to the generalized join shortest queue. in *Advances in Queueing Theory and Network Applications*, edited by Yue, W., Takahashi, Y. and Takaki, H., 3–33, Springer, New York.

[50] Miyazawa, M. (2009). Tail decay rates in double QBD processes and related reflected random walks. *Mathematics of Operations Research*, 34, 547–575.

[51] Miyazawa, M. (2011). Light tail asymptotics in multidimensional reflecting processes for queueing networks. *TOP*, 19(2), 233–299.

[52] Miyazawa, M., & Rolski, T. (2009). Tail asymptotics for a Lévey-driven tandem queue with an intermediate input. *Queueing Systems*, 63, 323–353.

[53] Miyazawa, M., & Zhao, Y. Q. (2004). The stationary tail asymptotics in the $GI/G/1$ type queue with countably many background states. *Advances in Applied Probability*, 36(4), 1231–1251.

[54] Morrison, J. A. (2007). Processor sharing for two queues with vastly different rates. *Queueing Systems*, 57, 19–28.

[55] Motyer, Allan J., & Taylor, Peter G. (2006). Decay rates for quasi-birth-and-death process with countably many phases and tri-diagonal block generators. *Advances in Applied Probability*, 38, 522–544.

[56] Raschel, K. (2010). Green functions and Martin compactification for killed random walks related to SU(3).*Electronic. Communications. in Probability*, 15, 176–190.

[57] Song, Y., Liu, Z., & Dai, H. (2015). Exact tail asymptotics for a discrete-time preemptive priority queue.*Acta Mathematicae Applicatae Sinica, English Series*, 31(1), 43–58.

[58] Song, Y., Liu, Z., & Zhao, Y. Q. (2016). Exact tail asymptotics-Revisit of a retrial queue with two input streams and two orbits. *Annals of Operations Research*, 247(1), 97-120.

[59] Takahashi, Y., Fujimoto, K., & Makimoto, N. (2001). Geometric decay of the steady-state probabilities in a quasi-birth-and-death process with a countable number of phases. *Stochastic Models*, 17(1), 1–24.

[60] Tang, J., & Zhao, Y. Q. (2008). Stationary tail asymptotics of a tandem queue with feedback. *Annals of Operations Research*, 160, 173-189.

[61] Wright, P. (1992). Two parallel processors with coupled inputs. *Advances in Applied Probability*, 24, 986–1007.

[62] Ye, W., Li, H., & Zhao, Y. Q. (2015). Tail behaviour for longest-queue-served-first queueing system-A random walk in the half plane. *Stochastic Models*, 31(3), 452–493.