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Sampling and Importance Resampling in Traffic Intensity Estimates in Markovian Single-server Queues

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Abstract: Markovian single-server queues are among the simplest models in queueing theory. Such queues have important practical applications. Essentially, it is of great interest to determine their traffic intensity, defined as the ratio between the arrival and service rates, which represents the fraction of time the queue is busy and allows for the calculation of other important performance measures, such as the average queue size and the expected number of people in the system. Fundamentally, the problem addressed here is how traffic intensity can be estimated, based on the number of arrivals during service time, using a popular Bayesian method known as sampling/importance sampling. The performance of the proposed estimators is analyzed for different values in the parametric space. Notably, it is observed that the Bayesian estimators are computationally feasible and superior to the classic maximum likelihood estimators in many situations. A numerical example is presented in detail to illustrate the developed procedures.

Keywords: Bayesian estimation, Markovian queues, SIR.

1. Introduction

Communication theory [18], computer design [16, 17], manufacturing processes [21, 12], healthcare systems [1, 34, 2], and transportation systems [33] are just a few examples of the applications of queueing theory. Even though system managers may not be very interested in stationary queues, basic steady-state models such as Markovian single-server queues, which are represented in Kendall notation by $M/M/1$, are of interest because they can be seen as the first step towards a more elaborate analytical process with more sophisticated queueing models. Once the appropriate model is selected, the next task is the statistical estimation of its parameters, which is the objective of many recent studies [for example, 20, 6, 7, 3, 25, 26, 27, 4, 28, 29, 30, 8, 9, and references therein].

The focus of this study is to propose point and interval estimations using the sample/importance resampling method (SIR) for the traffic intensity in $M/M/1$ queues, that is, the fraction of time the queue is busy, defined as the ratio between the arrival and service

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rates, based on the number of arrivals during service times. The SIR was successfully used in estimating parameters in Markovian single and multi-server queues, as reported by Cruz et al. [7]. Note that traffic intensity could be evaluated by estimating the arrival and service rates and then arriving at the corresponding traffic intensity, which is the ratio between these two quantities. However, this procedure requires samples of the time between arrivals in addition to customer service times. Thus, the collection process is more complex than simply collecting data on the number of arrivals during service time. Consequently, the collection cost is higher.

The remainder of this paper is organized as follows. Section 2 presents the mathematical foundation. Next, Section 3 discusses the results of the Monte Carlo simulations. Subsequently, Section 4 demonstrates a detailed numerical example. Finally, Section 5 presents the concluding remarks and topics for future research in this area.

2. Materials and Methods

Consider a Markovian single-server queue $M/M/1$. In the $M/M/1$ model, the number of arrivals Y follows a Poisson distribution with the following probability mass function (PMF):

$$P(Y = y) = \frac{e^{-\lambda t}(\lambda t)^y}{y!},\ y = 0, 1, \ldots, \tag{1}$$

and the time between departures follows an exponential distribution. Therefore, we have the following probability density function (PDF) for the time between departures,

$$b(t) = \mu e^{-\mu t}, \tag{2}$$

where $\lambda > 0$ and $\mu > 0$ are the arrival and service rates, respectively, and $\rho = \lambda/\mu$ is the traffic intensity. Assuming that the queue is stable (i.e., $0 < \rho < 1$) and that the equilibrium state has been reached, the following method of collecting data for the estimation of ρ is considered, with the respective PMF.

Suppose that $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$ are independent and identically distributed random variables, in which X_i denotes *the number of arrivals during the service of the i-th client*. Note that this sampling scheme is successfully used in [32]. Thus, considering that the random variables are independent, i may be ignored, resulting in the PMF of random variable X [13]

$$\begin{aligned} P(X = x) &= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^x}{x!}\mu e^{-\mu t} dt = \\ &= \left(\frac{\rho}{1+\rho}\right)^x \frac{1}{1+\rho},\ x = 0, 1, 2, \ldots, \end{aligned} \tag{3}$$

which may be recognized as a geometric distribution with parameter $1/(1+\rho)$, with $\rho = \lambda/\mu > 0$, known as traffic intensity.

2.1. Estimates by maximum likelihood

Suppose that a sample of size n is collected from the PMF given by Eq. (3); that is, $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$. Thus, the likelihood function, based on sample **x**, is given by

$$\begin{aligned} L(\rho|\mathbf{x}) &= (\frac{\rho}{1+\rho})^{x_1}(\frac{1}{1+\rho}) \times (\frac{\rho}{1+\rho})^{x_2}(\frac{1}{1+\rho}) \times \cdots \times (\frac{\rho}{1+\rho})^{x_n}(\frac{1}{1+\rho}) \\ &= (\frac{\rho}{1+\rho})^{y}(\frac{1}{1+\rho})^{n}, \end{aligned} \tag{4}$$

in which $y = \sum_{i=1}^{n} x_i$, which produces the following MLE for ρ,

$$\hat{\rho}_{\text{MLE}} = \arg\max_{\rho} L(\rho|\mathbf{x}) = \frac{y}{n}. \tag{5}$$

It can be seen that once the traffic intensity is known, other important performance measures can be found, such as the average queue size and the expected number of people in the system, through the following expressions [13]:

$$L_q = \frac{\rho^2}{1-\rho}, \tag{6}$$

and

$$L = L_q + \rho = \frac{\rho}{1-\rho}. \tag{7}$$

For the interval estimates of ρ, the bootstrap is proposed in its non-parametric version. Bootstrapping is a well-known computationally intensive technique proposed by Efron [10], where B resamplings with replacement $\mathbf{x}_{(i)}$ (with $B = 5,000$, which is typical) are drawn from the original sample **x** and the maximum likelihood estimates of traffic intensity are obtained for each of them, $\hat{\rho}_{\text{MLE}(i)}$. The bootstrap method is useful when the distribution of the parameter of interest is unknown [11], as in the case of ρ. The bootstrap algorithm for interval estimates of ρ based on the MLE is illustrated in Figure 1.

The option considered here is the use of percentiles 2.5% and 97.5% of the B bootstrap estimates $\hat{\rho}_{\text{MLE}(i)}$ to compute the lower limit (L) and upper limit (U), respectively, of the 95% confidence intervals; that is,

$$\begin{cases} L_{\text{MLE}} = \hat{\rho}_{\text{MLE}(2.5)}, \\ U_{\text{MLE}} = \hat{\rho}_{\text{MLE}(97.5)}. \end{cases} \tag{8}$$

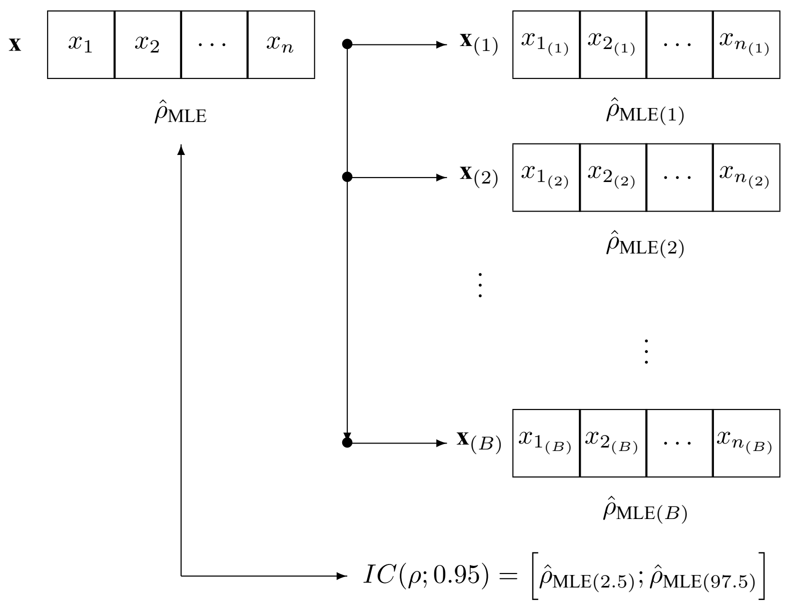


Figure 1. Bootstrap algorithm for the interval estimation of ρ

2.2. Bayesian estimates using SIR

The Bayesian approach is well known in statistical inference and is widely used in inference in queues [see, for instance, articles recently published by 28, 29, and references therein]. Statistical inference from a Bayesian viewpoint treats the parameter of interest ρ as a random variable and begins by defining a probability density function to model ρ in its parametric space Ω, called the prior distribution $\pi(\rho|\boldsymbol{\alpha})$, which conveniently models ρ through an appropriate choice of $\boldsymbol{\alpha}$, which is the vector of hyperparameters. After collecting a random sample of size n, $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$, the likelihood function $L(\rho|\mathbf{x})$ given by Eq. (4) can be defined, and Bayes' theorem can be used to obtain the posterior distribution of ρ as follows:

$$g\left(\rho|\mathbf{x}, \boldsymbol{\alpha}\right) = \frac{L(\rho|\mathbf{x}) \cdot \pi(\rho|\, \boldsymbol{\alpha})}{p(\mathbf{x}, \boldsymbol{\alpha})},\ \rho \in \Omega, \tag{9}$$

where Ω is the parametric space and $p(\mathbf{x}, \boldsymbol{\alpha}) = \int_{\rho\in\Omega} L(\rho|\mathbf{x}) \cdot \pi(\rho|\boldsymbol{\alpha}) d\rho$ is the normalization constant, which ensures that $g(\rho|\mathbf{x}, \boldsymbol{\alpha})$ is a probability density function, that is, $\int_{\rho\in\Omega} g(\rho|\mathbf{x}, \boldsymbol{\alpha})$ $d\rho = 1$. Thus, by using the posterior distribution given in Eq. (9), all inferences related to the parameter of interest ρ can be made. For instance, under the common squared error loss function [35], the Bayesian estimator of ρ can be obtained as $\hat{\rho} = \mathbb{E}(\rho|\mathbf{x}) = \int_{\rho\in\Omega} \rho g(\rho|\mathbf{x}, \boldsymbol{\alpha}) d\rho$.

The non-interactive *sampling/importance resampling* SIR method, discussed in detail by Rubin [24] and Smith and Gelfand [31], is proposed to perform Bayesian inference. The SIR method has the advantage of not requiring the resolution of integrals such as $p(\mathbf{x}, \boldsymbol{\alpha})$, which may not have a trivial solution. The SIR method is described as follows.

A random sample of size k is selected from the prior probability density function $\pi(\rho|\boldsymbol{\alpha})$, where $k \geq 5,000$. The sample values are denoted by $\rho_{\text{prior},i}, i = 0, 1, \ldots, k$. A weight function $W_i, i = 0, 1, \ldots, k$ is calculated for each point $\rho_{\text{prior},i}$, where W_i is proportional to the likelihood function in Eq. (4). Finally, a random sample of size k is selected from the prior distribution $\rho_{\text{prior},i}, i = 0, 1, \ldots, k$, with probabilities proportional to W_i. This new sample, called $\rho_{\text{post},i}, i = 0, 1, \ldots, k$, can be considered as coming from the posterior distribution of ρ and can be used for the Bayesian estimates of ρ, that is, as an estimate of $\mathbb{E}(\rho|\mathbf{x})$, through the following expression:

$$\hat{\rho} = \frac{\sum_{i=1}^{k} \rho_{\text{post},i}}{k}. \tag{10}$$

However, Ross [23] describes a more efficient implementation of the SIR method, in which resampling of the prior distribution need not be performed to obtain the posterior sample and, consequently, to estimate the parameter ρ. Essentially, the SIR method is based on estimating the amount $\mathbb{E}(\rho|\,\mathbf{x})$ using Eq. (10). However, Ross argues that as per the Rao-Blackwell theorem, the expression

$$\hat{\rho}_{\text{SIR}} = \frac{\sum_{i=1}^{k} W_i \cdot \rho_{\text{prior},i}}{\sum_{i=1}^{k} W_i} \tag{11}$$

has the same mean and a smaller variance than Eq. (10) and therefore has a lower mean square error when estimating ρ. Thus, it is preferred that such an estimate is made using Eq. (11), where W_i is proportional to the likelihood function in Eq. (4) and $\rho_{\text{prior},i}$ were sampled from the prior distribution. The algorithm for the Bayesian point estimates by SIR is shown in Figure 2.

algorithm
/* select a sample of size $k = 5,000$ from the prior distribution */
$\rho_{\text{prior},i} \sim \pi(\rho|\alpha),\ i = 1, \ldots, k$
/* for each $\rho_{\text{prior},i}$ and a given sample **x**, compute the weights W_i */
$W_i \leftarrow L(\rho_{\text{prior},i}|\mathbf{x}),\ i = 1, \ldots, k$
/* compute SIR estimate */
$$\hat{\rho}_{\text{SIR}} = \frac{\sum_{i=1}^{k} W_i \cdot \rho_{\text{prior},i}}{\sum_{i=1}^{k} W_i}$$
end algorithm

Figure 2. SIR algorithm for the point estimates of ρ

Similar to the confidence intervals obtained from the MLE, their Bayesian counterparts, called credible intervals, can be obtained from the 2.5% and 97.5% percentiles of the posterior estimates $\rho_{\text{post},i}$ for the lower and upper limits, respectively, which gives

$$\begin{cases} L_{\text{SIR}} = \rho_{\text{post},(2.5)}, \\ U_{\text{SIR}} = \rho_{\text{post},(97.5)}. \end{cases} \tag{12}$$

3. Computational Results

Computational experiments were carried out based on Monte Carlo simulations, in accordance with the algorithm presented in Figure 3. The experiments sought to establish, through the mean estimation errors, their variances, and the root mean square errors (REQMs), the effectiveness of the SIR method. For this, samples of size $n \in \{10, 20, 50, 80, 100, 200\}$ were used, with 1,000 Monte Carlo replications, which is a typical value. The samples were generated from a geometric distribution with parameter $1/(1+\rho)$, with $\rho \in \{0.1, 0.2, 0.4, 0.5, 0.6, 0.8, 0.9\}$. The sample size generated from the prior distribution of ρ is fixed at $k = 5,000$ and, for the boostrap, $B = 5,000$ resamplings, which is typically used. For comparison in different situations, three different types of prior distributions were considered: (i) a uniform distribution, representing a situation in which the data analyst has no prior information about the values of the parameters ρ; (ii) a triangular distribution with a mode located exactly at the simulated value of the parameter (which is unknown in practice), representing a situation in which the analyst has the best possible prior information about the location of the parameter; and (iii) a triangular distribution with a mode located at a point distant from the simulated values of the parameter (unknown in practice), representing a situation in which the analyst has the worst possible prior information about the location of the parameter. The estimators were programmed in R [22], and the code used is available directly from the authors on request.

algorithm
  /* perform Monte Carlo simulation */
 for $i = 1$ **until 1,000 do**
  /* generate a random sample of size n */
  $\mathbf{x}_i \leftarrow \{x_{i,1,}, x_{i,2}, \ldots, x_{i,n}\}$, $x_{i,j} \leftarrow \text{Geo}\left(\frac{1}{1+\rho}\right)$, $j = 1, \ldots, n$
  /* estimate parameter by chosen method */
  $\hat{\rho}_i \leftarrow \text{Estimator}(\mathbf{x}_i)$
 end for
  /* write results */
 write $\text{Mean}(\hat{\rho}_i), \text{Var}(\hat{\rho}_i)$
end algorithm

Figure 3. Monte Carlo algorithm for the evaluation of the estimators

3.1. Performance of point estimates

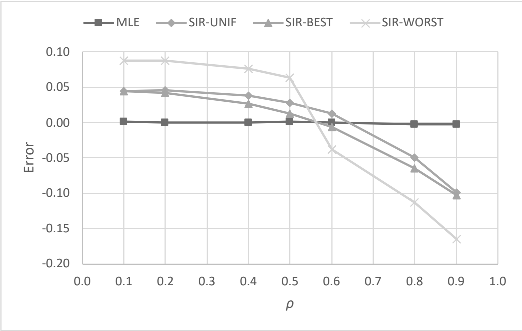
Table 1 presents the computational results obtained for the point estimators of the traffic intensity ρ, using the MLE and SIR methods, for different prior distributions, through values of the averages and variances of the estimates. It should be noted that, in general, the estimates are close to the simulated values and the variances are quite small, as is desirable.

Figure 4 summarizes the previous table, in terms of the average performance of the estimators, as a function of the position of the parameter ρ in the parametric space and as a function of the sample size. Note that although the MLE generally presents the smallest

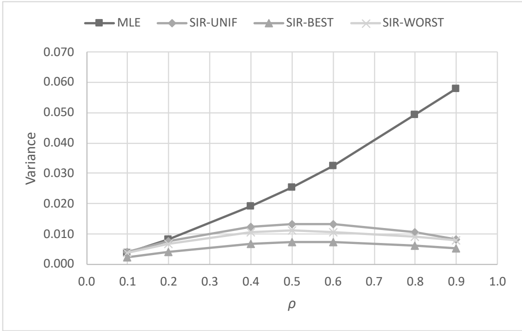
Table 1. Computational results for the point estimators of ρ

		Average				Variance			
n	ρ	MLE	SIR-UNIF	SIR-BEST	SIR-WORST	MLE	SIR-UNIF	SIR-BEST	SIR-WORST
10	0.1	0.1045	0.2372	0.2200	0.3701	0.0114	0.0121	0.0052	0.0096
	0.2	0.2067	0.3332	0.3091	0.4535	0.0259	0.0204	0.0085	0.0150
	0.4	0.4057	0.4810	0.4510	0.5757	0.0606	0.0258	0.0108	0.0171
	0.5	0.5097	0.5422	0.5134	0.6250	0.0796	0.0246	0.0105	0.0158
	0.6	0.6099	0.5918	0.5691	0.4925	0.1007	0.0227	0.0100	0.0183
	0.8	0.8075	0.6666	0.6692	0.5636	0.1546	0.0199	0.0097	0.0178
	0.9	0.9070	0.6975	0.7184	0.5941	0.1825	0.0175	0.0094	0.0166
20	0.1	0.1006	0.1668	0.1695	0.2344	0.0059	0.0071	0.0040	0.0076
	0.2	0.1971	0.2706	0.2662	0.3404	0.0123	0.0137	0.0073	0.0138
	0.4	0.3968	0.4625	0.4399	0.5256	0.0288	0.0213	0.0109	0.0188
	0.5	0.4974	0.5424	0.5148	0.5989	0.0386	0.0215	0.0110	0.0178
	0.6	0.5965	0.6093	0.5818	0.5315	0.0493	0.0198	0.0103	0.0154
	0.8	0.7897	0.7089	0.6972	0.6236	0.0738	0.0155	0.0088	0.0136
	0.9	0.8889	0.7479	0.7520	0.6617	0.0868	0.0128	0.0080	0.0121
50	0.1	0.1012	0.1263	0.1312	0.1513	0.0022	0.0024	0.0018	0.0027
	0.2	0.2005	0.2299	0.2326	0.2558	0.0045	0.0049	0.0036	0.0051
	0.4	0.3982	0.4342	0.4260	0.4640	0.0107	0.0110	0.0072	0.0112
	0.5	0.4992	0.5339	0.5174	0.5637	0.0141	0.0129	0.0079	0.0126
	0.6	0.5985	0.6231	0.6003	0.5748	0.0181	0.0131	0.0077	0.0099
	0.8	0.7971	0.7629	0.7413	0.6998	0.0272	0.0093	0.0057	0.0079
	0.9	0.8962	0.8119	0.8023	0.7469	0.0323	0.0067	0.0045	0.0063
80	0.1	0.1012	0.1166	0.1203	0.1314	0.0014	0.0014	0.0011	0.0014
	0.2	0.2008	0.2188	0.2215	0.2347	0.0030	0.0031	0.0025	0.0032
	0.4	0.4028	0.4258	0.4220	0.4444	0.0068	0.0071	0.0053	0.0073
	0.5	0.5027	0.5272	0.5172	0.5467	0.0092	0.0093	0.0063	0.0094
	0.6	0.6042	0.6259	0.6081	0.5911	0.0118	0.0103	0.0066	0.0080
	0.8	0.8032	0.7843	0.7617	0.7322	0.0168	0.0073	0.0046	0.0061
	0.9	0.9031	0.8392	0.8260	0.7846	0.0200	0.0049	0.0033	0.0046
100	0.1	0.0993	0.1116	0.1150	0.1237	0.0011	0.0012	0.0010	0.0011
	0.2	0.1983	0.2126	0.2153	0.2250	0.0025	0.0026	0.0021	0.0026
	0.4	0.3973	0.4156	0.4138	0.4303	0.0060	0.0063	0.0049	0.0064
	0.5	0.4961	0.5160	0.5097	0.5316	0.0078	0.0079	0.0057	0.0080
	0.6	0.5944	0.6135	0.6008	0.5860	0.0102	0.0095	0.0063	0.0075
	0.8	0.7919	0.7804	0.7608	0.7349	0.0158	0.0078	0.0050	0.0064
	0.9	0.8900	0.8402	0.8277	0.7908	0.0182	0.0051	0.0036	0.0047
200	0.1	0.0997	0.1058	0.1077	0.1117	0.0005	0.0006	0.0005	0.0006
	0.2	0.1995	0.2066	0.2082	0.2127	0.0012	0.0012	0.0011	0.0012
	0.4	0.4001	0.4091	0.4089	0.4164	0.0028	0.0028	0.0024	0.0029
	0.5	0.4993	0.5094	0.5073	0.5171	0.0037	0.0037	0.0031	0.0038
	0.6	0.5986	0.6096	0.6040	0.5960	0.0046	0.0047	0.0036	0.0041
	0.8	0.7973	0.7984	0.7820	0.7670	0.0069	0.0048	0.0031	0.0038
	0.9	0.8969	0.8698	0.8557	0.8324	0.0082	0.0030	0.0021	0.0027

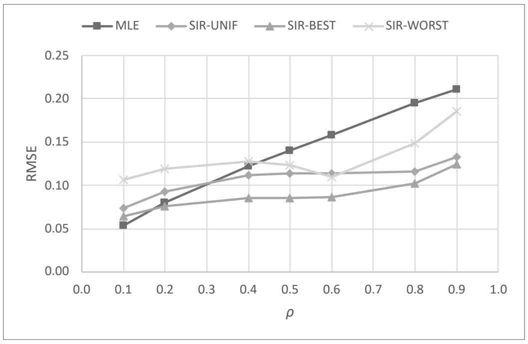
mean errors, as seen in Figure 4-(a), its variances are the largest among all, as shown in Figure 4-(b). In addition, the Bayesian estimators have an equivalent performance in terms of the average error, for high values of traffic intensity ($\rho \geq 0.5$), combined with a smaller variance, both for a uniform vague prior distribution and for the best case of the triangular prior, indicating that this method is preferable in this situation. Under a bad prior (the worst case of the triangular one), the estimator's variance is equivalent to the MLE. However, it presents the worst error among all the estimators, indicating that in the elicitation of a prior distribution, the analyst needs to present good information or choose a vague prior, that is, opt for a uniform prior distribution. Regarding the sample size n, the errors, as shown in Figure 4-(c), and the mean variances, as seen in Figure 4-(d), reach zero as n increases, which is encouraging.



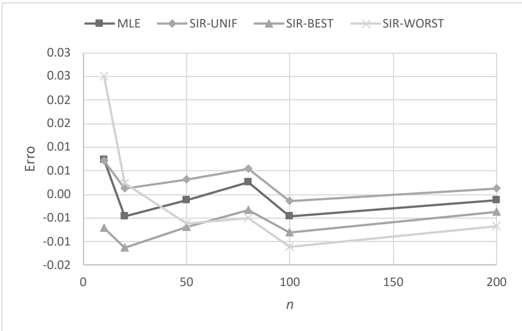
(a) mean errors for ρ



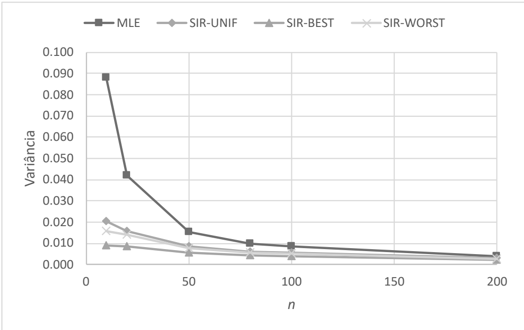
(b) mean variances for ρ



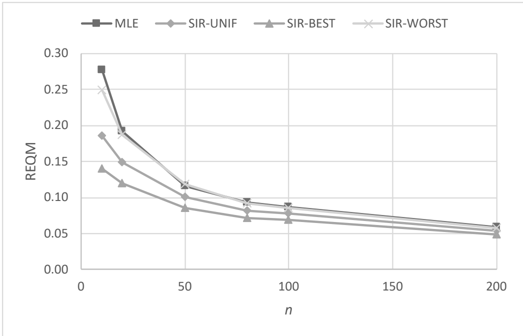
(c) mean RMSEs for ρ



(d) mean errors for n



(e) mean variances for n



(f) mean RMSEs for n

Figure 4. Performance of the point estimators of ρ

3.2. Performance of interval estimates

Table 2 presents the width and coverage of the 95% interval estimates of ρ by employing the MLE and SIR methods for the three different prior distributions used earlier. From the results presented, we can see that, although the MLE may have the smallest widths, its coverage may be worse than that of the Bayesian method, except when the prior distribution is the worst possible. In addition, all methods yield nominal coverage of 95% as n increases.

Table 2 is summarized by Figure 5, which presents the averages of the widths and coverage as functions of ρ, as seen in Figures 5-(a) and -(b), and n, as shown in Figures 5-(c) and

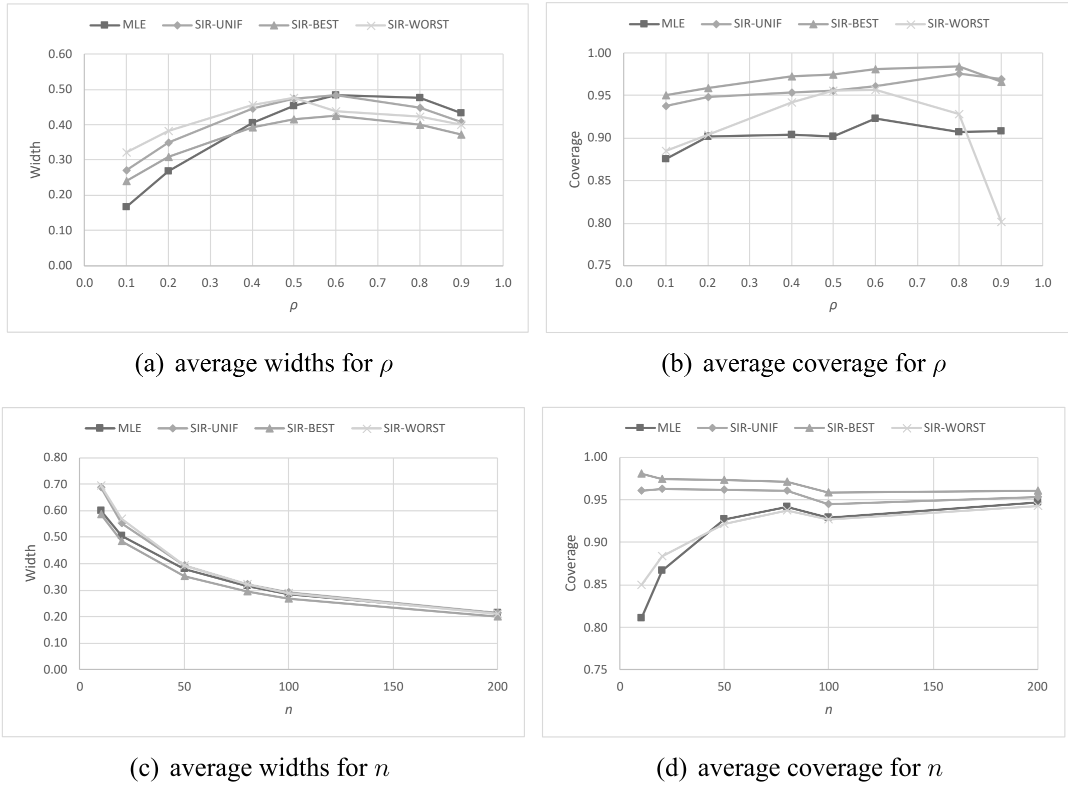
Table 2. Computational results for the interval estimators of ρ

		Width				Coverage			
n	rho	MLE	SIR-UNIF	SIR-BEST	SIR-WORST	MLE	SIR-UNIF	SIR-BEST	SIR-WORST
10	0.1	0.2601	0.6449	0.5196	0.7904	0.620	0.904	0.930	0.742
	0.2	0.4499	0.7157	0.5840	0.7878	0.824	0.938	0.973	0.833
	0.4	0.6636	0.7423	0.6244	0.7409	0.851	0.972	0.994	0.948
	0.5	0.7151	0.7300	0.6221	0.7094	0.825	0.969	1.000	0.976
	0.6	0.7286	0.7061	0.6104	0.6344	0.888	0.986	1.000	0.957
	0.8	0.7087	0.6501	0.5751	0.6112	0.827	0.980	0.994	0.891
	0.9	0.6807	0.6214	0.5568	0.5969	0.836	0.972	0.972	0.601
20	0.1	0.2250	0.3896	0.3495	0.4948	0.836	0.938	0.938	0.851
	0.2	0.3681	0.5161	0.4435	0.5959	0.867	0.969	0.969	0.872
	0.4	0.5587	0.6313	0.5339	0.6532	0.860	0.960	0.988	0.935
	0.5	0.6094	0.6352	0.5436	0.6360	0.860	0.970	0.988	0.963
	0.6	0.6283	0.6194	0.5380	0.5558	0.893	0.982	0.997	0.968
	0.8	0.5950	0.5573	0.4990	0.5295	0.873	0.970	0.985	0.892
	0.9	0.5588	0.5237	0.4759	0.5106	0.874	0.953	0.954	0.707
50	0.1	0.1666	0.2037	0.1955	0.2308	0.943	0.961	0.973	0.916
	0.2	0.2542	0.2941	0.2760	0.3179	0.916	0.946	0.957	0.923
	0.4	0.3936	0.4367	0.3884	0.4578	0.919	0.956	0.972	0.943
	0.5	0.4526	0.4810	0.4193	0.4931	0.921	0.958	0.975	0.963
	0.6	0.4896	0.4923	0.4292	0.4409	0.934	0.961	0.985	0.957
	0.8	0.4711	0.4422	0.3982	0.4198	0.926	0.982	0.982	0.926
	0.9	0.4207	0.3992	0.3670	0.3942	0.928	0.970	0.966	0.823
80	0.1	0.1395	0.1570	0.1517	0.1682	0.947	0.939	0.952	0.926
	0.2	0.2090	0.2283	0.2180	0.2395	0.933	0.957	0.962	0.937
	0.4	0.3210	0.3450	0.3180	0.3568	0.941	0.955	0.974	0.950
	0.5	0.3708	0.3921	0.3520	0.4025	0.944	0.958	0.972	0.952
	0.6	0.4133	0.4194	0.3698	0.3793	0.944	0.957	0.978	0.961
	0.8	0.4097	0.3855	0.3470	0.3646	0.943	0.984	0.986	0.953
	0.9	0.3568	0.3397	0.3133	0.3370	0.938	0.977	0.972	0.883
100	0.1	0.1263	0.1391	0.1349	0.1463	0.952	0.946	0.957	0.933
	0.2	0.1889	0.2025	0.1947	0.2102	0.928	0.936	0.945	0.926
	0.4	0.2897	0.3074	0.2873	0.3159	0.914	0.934	0.950	0.930
	0.5	0.3344	0.3524	0.3212	0.3603	0.925	0.930	0.956	0.933
	0.6	0.3743	0.3831	0.3413	0.3496	0.930	0.927	0.961	0.939
	0.8	0.3742	0.3561	0.3220	0.3376	0.930	0.971	0.978	0.947
	0.9	0.3206	0.3103	0.2877	0.3098	0.926	0.970	0.964	0.881
200	0.1	0.0911	0.0954	0.0931	0.0970	0.957	0.939	0.951	0.942
	0.2	0.1347	0.1394	0.1358	0.1421	0.942	0.942	0.946	0.936
	0.4	0.2067	0.2122	0.2041	0.2150	0.942	0.945	0.957	0.946
	0.5	0.2389	0.2452	0.2331	0.2482	0.938	0.948	0.960	0.946
	0.6	0.2704	0.2767	0.2573	0.2623	0.952	0.956	0.963	0.957
	0.8	0.3040	0.2917	0.2623	0.2721	0.947	0.968	0.978	0.959
	0.9	0.2601	0.2500	0.2315	0.2465	0.949	0.978	0.972	0.915

-(d). The figure shows that the SIR method presents greater average width than the MLE method for $\rho < 0.4$. However, for $n \geq 100$, the average widths are approximately the same for all methods. Furthermore, the Bayesian method's average coverage is superior and closer to 95% along the parametric space, with all methods converging to the nominal 95% as n increases, as expected.

4. Numerical Example

This section exemplifies the methodology using a practical situation to facilitate the reader's understanding. Consider the following example: a vaccine post is assumed for vaccination against the COVID-19 pandemic, which currently represents a major global health



(a) average widths for ρ

(b) average coverage for ρ

(c) average widths for n

(d) average coverage for n

Figure 5. Performance of the interval estimators of ρ

concern [14]. A factor that managers will have to control in such a situation is the number of people in the post since a large number of people in the same place generates a higher risk of contamination.

We sought to determine whether it would be necessary to control the number of people in a vaccine administration site. Having a professional to control the number of people at all vaccination sites in the country is economically unfeasible. To this end, we must first consult a health professional on the number of people who can be present at each site.

In this example, we consider a specific health center. Based on several previous studies, we can say that people arrive at the center during a given period of time following a Poisson distribution and that the service times are exponentially distributed. According to a previously consulted health professional, this site can host a maximum of six people without a high risk of contamination.

To determine whether it is necessary to control the entry of people in this system (which did not have entry control), we followed the following steps.

Step 1: Using the PMF for the number X of users present in the system at random times [13],

$$P(X = x) = \rho^x(1-\rho),\ x = 0, 1, 2, \dots, \tag{13}$$

we will numerically derive the acceptable arrival rate that guarantees $P(X \leq 6) > 95\%$, that is:

$$\sum_{k=0}^{6} P(X = k) \geq 0.95 \therefore \rho \leq 0.651836.$$

Step 2: We define a null hypothesis H_0, where $\rho \leq 0.651836$, and an alternative hypothesis H_1, where $\rho > 0.651836$, that is:

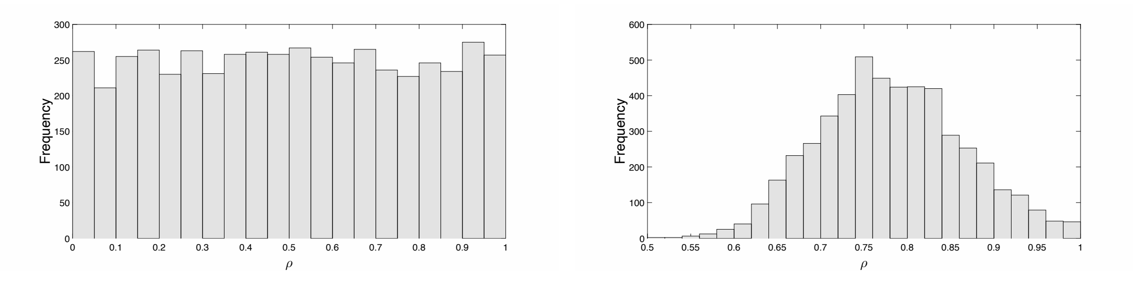
$$\begin{cases} H_0 : \rho \leq 0.651836, \\ H_1 : \rho > 0.651836. \end{cases}$$

Step 3: A sample was collected for the number of arrivals during 200 consecutive services, presented in Table 3, in which the value 0 (zero) is observed 112 times, the value 1 (one) is observed 51 times, and so on.

Table 3. Observed values (O) and their frequencies (F) for a random sample of size $n = 200$

	number of arrivals x							
0 →	0	1	2	3	4	5	6	> 6
F →	112	51	20	8	7	1	1	0

Step 4: Once the data were collected, ρ was estimated via the SIR method, under a uniform prior distribution, since no previous information regarding the vaccine post was available. The posterior distribution was obtained by presenting the average $\hat{\rho} \simeq 0.7813728$. The histograms of these two distributions are shown in Figure 6.



(a) prior distribution (b) posterior distribution

Figure 6. Histograms for prior and posteriors distributions

Step 5: Using the prior and posterior distributions, the hypothesis test defined in step 3 can be performed. According to Choudhury and Borthakur [5], the Bayes factor (BF) can be determined using the following expression:

$$\frac{P(H_1|\text{data})}{P(H_0|\text{data})} = \text{FB} \frac{P(H_1)}{P(H_0)},$$

in which:

- $P(H_1|\text{data})$ represents the percentage of values in the posterior distribution that fit hypothesis H_1, that is, those that are greater than 0.651836;
- $P(H_0|\text{data})$ represents the percentage of values in the posterior distribution that fit hypothesis H_0, that is, those that are less than or equal to 0.651836;
- $P(H_1)$ represents the percentage of values in the prior distribution that fit hypothesis H_1, that is, those that are greater than 0.651836;
- $P(H_0)$ represents the percentage of values in the prior distribution that fit hypothesis H_0, that is, those that are less than or equal to 0.651836.

Using the prior and posterior distributions generated earlier, we find that FB $\simeq 34.4112$.

Step 6: Finally, we used the rule defined by Kass and Raftery [15], when choosing between H_0 and H_1, according to Table 4.

Table 4. Rule when choosing between H_0 e H_1 [15]

$2\ln(\text{FB})$	Evidence against H_0
0 a 2	Minimal
2 a 6	Positive
6 a 10	Strong
> 10	Decisive

Step 7: The calculation of $2\ln(\text{BF})$ results in $\simeq 7.076764$. Thus, we can say that there is strong evidence against H_0; that is, there is decisive evidence against the hypothesis that there will be no health risks due to too many people being at this vaccine center, making it necessary to control the entry of people.

5. Conclusions and Final Remarks

From the results obtained in Section 3, we can conclude that the non-interactive method SIR proved to be, in general, superior to the MLE. Furthermore, as the sample size increased, all estimators performed similarly and converged to the simulated value. Regarding the confidence intervals, although the widths of the intervals obtained by the MLE were smaller than those obtained by the SIR, the coverage of the intervals obtained by the SIR method was better because they were closer to the nominal value (95%) throughout the entire parametric space.

Further, if the prior distribution chosen for the SIR method represents the real value of ρ, the Bayesian estimator of ρ proves superior throughout the parametric space, which is a possible and a practical situation. In many systems, workers or system managers have a reasonable idea of the value of traffic intensity.

There are several future research directions. Future research can verify the efficiency of other sampling methods, such as counting the number of arrivals during the service of n users. Another alternative would be to use the number of users present in the system at random times. In addition, new research can be developed for estimations of other types of queues, such as multi-server Markov queues (or $M/M/s$ queues in Kendall's notation) and finite Markov queues ($M/M/1/k$).

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Declarations

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Conflicts of Interest/Competing Interests

The funder had no role in the study design, data collection and analysis, publication decision, or manuscript preparation. The authors declare no conflicts of interest regarding the publication of this article.

Availability of Data and Materials

The data used to support the findings of this study have been included in this article.

Code Availability

The proposed algorithms can be encoded in the reader's choice of programming language. R scripts can be obtained from the authors upon request.

Authors' Contributions

VBQ, FRBC, and RCQ contributed equally to the design and implementation of the research, analysis of the results, and final writing of the manuscript.

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