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On a Geo/G/1 Queue with Disastrous and Non-disastrous Failures

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Abstract: This paper considers a Geo/G/1 queue system with disastrous and non-disastrous failures. In the discussed system, once the server suffers from a disastrous failure, all customers in the system are lost at once and the server is immediately sent to a repair facility for fixing. During the repair period of a disastrous failure, arrival customers are blocked and turned away till the server has been repaired. The server may also encounter a non-disastrous failure. During the repair time of a non-disastrous failure, arriving customers are allowed to join the system and in the meanwhile a disastrous failure may occur. By using supplementary variable technique, some system characteristics such as the idle period and the busy cycle are derived. The system size and the expected sojourn time are also obtained. Finally, some numerical examples and results are presented.

Keywords: Disastrous failure, non-disastrous failure, sojourn time, supplementary variable technique, system size.

1. Introduction

In the works of queueing theory, although several publications discussed the unreliable server problem, they usually targeted on customers that have never been lost and continued to enter into the system. Only few works considered the case where customers are blocked and removed from the system. Such a queueing system is called ‘queueing system with disasters’. Queueing systems with disasters have been intensively studied because of their great applications in complex modern communication systems, networks and manufacturing systems. Interested readers can refer to [1–8] and references therein.

Recently, there is a growing interest in the analysis of discrete-time queues with disasters because the discrete-time queue is more suitable for describing the telecommunication network, digital communication systems and other related areas. For

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example, Atencia and Moreno [9] investigated a Geo/Geo/1 queue with negative customers and disasters. They obtained the explicit expressions for the stationary distributions of the queue length and the system size. Yi *et al.* [10] analyzed a Geo/G/1 queue with disasters and multiple working vacations. For such a queueing system, they studied the queue length and derived the stationary queue length distribution. Park *et al.* [11] considered a Geo/G/1 queue with negative customers and disasters. They obtained the probability generating functions (PGFs) of the stationary queue length and the sojourn time of a customer. Moreover, Park *et al.* [12] analyzed a GI/Geo/1 queue with disasters, in which the inter-arrival times of customers follow a general distribution. Furthermore, Lee *et al.* [13] discussed a Geo/G/1 queue with disasters and general repair time. They derived the PGFs of the queue length distribution and the first-come first-serve sojourn time distribution. To analyze the power saving scheme in wireless sensor networks under unreliable network connections, Lee and Yang [14] studied a discrete-time Geo/G/1 queue with N-policy and disasters. They obtained the PGFs of the queue length, the sojourn time, and the regeneration cycles such as the idle period and the busy period. Jeyakumar and Gunasekaran [15] investigated a discrete queue with disaster and single vacation. They used generating function technique to find the PGFs of the vacation period, the idle period, and the busy period. The mean queue length was also obtained.

However, in practice, except for suffering from a disastrous failure, the server may also encounter a non-disastrous failure, which is caused by a normal breakdown at the same system. During the repair process of a non-disastrous failure, arriving customers are allowed to join the system and in the meanwhile a disastrous failure may also occur. To the best of our knowledge, no works consider a Geo/G/1 queueing system, where the server may suffer from disastrous and non-disastrous failures at the same system. This motivates us to consider a Geo/G/1 queue system with disastrous and non-disastrous failures. Besides, the discussed system can be applied to an application in computer network center with virus infection. In general, customers arrive at the network center when the computers are not affected by virus infection. When the computers are affected by a virus (disastrous failures), no customers can be allowed to enter into the network center and all the customers in the network center are removed from the system until all the computers are recovered from the virus infection. The recovery time of computers can be regarded as repair time. Besides, the computers may also suffer from a normal breakdown (non-disastrous failure). During the repair process of a non-disastrous failure, arriving customers are allowed to join the system and in the meanwhile a disastrous failure may also occur. The above process can be modeled as a Geo/G/1 queueing system with disastrous and non-disastrous failures.

The rest of this paper is organized as follows. Section 2 describes the investigated model. The analysis of the discussed system is presented in Section 3. Some performance measures are derived in Section 4. Numerical examples are provided in Section 5. Section 6 concludes the paper.

2. Model Description

In discrete-time queueing models, the time axis is divided into fixed-length intervals, called slots. It is assumed that customer arrivals and departures take place only at slot boundaries. This study deals with a Geo/G/1 queueing system subject to disasters and breakdowns. Customer arrivals are according to Bernoulli process. The probability of a customer arrival during any slot is λ (0<λ<1). Service times $\{B_n\}_{n=1}^{\infty}$ are independent and identically distributed discrete random variables with probability mass function (PMF) $P\{B_n = i\} = b_i$, $(i = 1,2,...)$, probability generating function (PGF) $B(x)$, and the j-th factorial moment $B^{(j)}$. The server may suffer from disastrous and non-disastrous failures. Once a disastrous failure occurs, the system removes all workloads and the server is sent to repair immediately. Note that a disastrous failure cannot occur when the server is idle. During the repair period of a disastrous failure, arriving customers are prohibited from entering into the system. The server may also encounter a normal breakdown (non-disastrous failure) and it is sent to repair immediately. During the repair process of a non-disastrous failure, customers can continue to arrive at the system and in the meanwhile a disastrous failure may also occur. The server will go to an idle state after the repair completion of a disastrous failure. Customer arrivals form a single waiting line based on the first-come first-serve (FCFS) discipline. The server can serve only one customer at a time. Customer arrivals entering into the service system must wait in the queue until their services are completed unless a disastrous failure occurs.

The distributions of disastrous and non-disastrous failures are geometrically distributed with parameters β (0<β<1) and α (0<α<1), respectively. The repair times of a disastrous failure $\{D_n\}_{n=1}^{\infty}$ are generally distributed with PMF $P\{D_n = i\} = d_i$ $(i = 1,2,...)$, PGF $D(x)$, and the j-th factorial moment $D^{(j)}$. The repair times of a non-disastrous failure $\{R_n\}_{n=1}^{\infty}$ are independent and identically distributed discrete random variables with PMF $P\{R_n = i\} = r_i$ $(i = 1,2,...)$, PGF $R(x)$, and the j-th factorial moment $R^{(j)}$. In discrete-time queueing models, different assumptions can be made on the order of arrivals and departures simultaneously taking place at a slot boundary: either arrivals may have precedence over departures or vice versa. The former case is referred to as the Late Arrival System (LAS) and the latter as the Early Arrival System (EAS). There are two variants of LAS, LAS with immediate access and LAS with delayed access. The difference between them is when a customer arrives late in the nth slot while the system is empty, the service is started in the nth slot (LAS with immediate access) or the service is started in the (n+1)-th slot (LAS with delayed access). We adopt the LAS policy with delayed access in the presented model. According to LAS policy, customer arrivals and departures occur within (t^-, t) and (t, t^+), respectively. We assume that the events of a disastrous failure, a non-disastrous failure, and a customer arrival occur within (t^-, t). The order of them is that a disastrous failure takes place just prior to a non-disastrous failure, and a non- non-disastrous failure takes place just prior to a customer arrival. A disastrous failure and a repair completion do not occur at the same slot boundary.

3. Mathematical Model

In the investigated system, let $\Gamma^{(n)}$ indicate the state of the system at time n^+ and $L^{(n)}$ denote the systems size, where

$$\Gamma^{(n)} = \begin{cases} 0, \text{ if the server is idle;} \\ 1, \text{ if the server is busy;} \\ 2, \text{ if the server is under repair times of a disastrous failure;} \\ 3, \text{ if the server is under repair times of a non-disastrous failure.} \end{cases}$$

When $\Gamma^{(n)} = 1$, $\varepsilon^{(n)}$ represents the remaining service time of a customer being served. If $\Gamma^{(n)} = 2$, $\varepsilon^{(n)}$ represents the remaining repair time of a disastrous failure. If $\Gamma^{(n)} = 3$, $\varepsilon^{(n)}$ represents remaining service time of a customer being served just before a non-disastrous failure occurring and $\xi^{(n)}$ corresponds to the remaining repair time of a non-disastrous failure. The sequence of $Y = \{(\Gamma^{(n)}, L^{(n)}, \varepsilon^{(n)}, \xi^{(n)}); n = 0,1,2,\cdots\}$ is a Markov chain whose state space is

$$\{(0,0)\} \bigcup \{(1,k,i) : k \ge 1, i \ge 1\} \bigcup \{(2,0,i) : i \ge 1\} \bigcup \{(3,k,i,j) : k \ge 1,\ i \ge 1, j \ge 1\}.$$

Next, we define the following steady state probabilities:

$$p_{0,0} = \lim_{n\to\infty} \Pr[\Gamma^{(n)} = 0, L^{(n)} = 0],$$

$$p_{1,k,i} = \lim_{n\to\infty} \Pr[\Gamma^{(n)} = 1, L^{(n)} = k, \varepsilon^{(n)} = i], \quad k \ge 1,\ i \ge 1,$$

$$p_{2,i} = \lim_{n\to\infty} \Pr[\Gamma^{(n)} = 2, L^{(n)} = 0, \varepsilon^{(n)} = i], \ \ i \ge 1,$$

$$p_{3,k,i,j} = \lim_{n\to\infty} \Pr[\Gamma^{(n)} = 2, L^{(n)} = k, \varepsilon^{(n)} = i, \xi^{(n)} = j], \quad k \ge 1,\ i \ge 1,\ j \ge 1.$$

The Kolmogorov equations for the stationary distribution are given by

$$p_{0,0} = p_{0,0}\bar{\lambda} + p_{1,1,1}\bar{\beta}\bar{\lambda} + p_{2,1}, \tag{1}$$

$$p_{1,1,i} = p_{0,0}\lambda\bar{\alpha}b_i + p_{1,1,1}\lambda\bar{\beta}\bar{\alpha}b_i + p_{1,1,i+1}\bar{\lambda}\bar{\beta}\bar{\alpha} + p_{1,2,1}\bar{\lambda}\bar{\beta}\bar{\alpha}b_i + p_{3,1,i,1}\bar{\lambda}\bar{\beta}\bar{\alpha},\ i \ge 1, \tag{2}$$

$$\begin{aligned} p_{1,k,i} &= p_{1,k-1,i+1}\lambda\bar{\beta}\bar{\alpha} + p_{1,k,1}\lambda\bar{\beta}\bar{\alpha}b_i + p_{1,k+1,1}\bar{\lambda}\bar{\beta}\bar{\alpha}b_i + p_{1,k,i+1}\bar{\lambda}\bar{\beta}\bar{\alpha} \\ &\quad + p_{3,k-1,i,1}\lambda\bar{\beta}\bar{\alpha} + p_{3,k,i,1}\bar{\lambda}\bar{\delta}\bar{\alpha}, \qquad k \ge 2,\ i \ge 1; \end{aligned} \tag{3}$$

$$p_{2,i} = p_{2,i+1} + \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} p_{1,\ell,m}\beta d_i + \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{3,\ell,m,n}\beta d_i, \tag{4}$$

$$\begin{aligned} p_{3,1,i,j} &= p_{0,0}\lambda\alpha b_i r_j + p_{1,1,1}\lambda\bar{\beta}\alpha b_i r_j + p_{1,1,i+1}\bar{\lambda}\bar{\beta}\alpha r_j + p_{1,2,1}\bar{\lambda}\bar{\beta}\alpha b_i r_j \\ &\quad + p_{3,1,i,1}\bar{\lambda}\bar{\beta}\alpha r_j + p_{3,1,i,j+1}\bar{\lambda}\bar{\beta}, \qquad i \ge 1,\ j \ge 1, \end{aligned} \tag{5}$$

$$\begin{aligned} p_{3,k,i,j} &= p_{1,k-1,i+1}\lambda\bar{\beta}\alpha r_j + p_{1,k,1}\lambda\bar{\beta}\alpha b_i r_j + p_{1,k,i+1}\bar{\lambda}\bar{\beta}\alpha r_j + p_{1,k+1,1}\bar{\lambda}\bar{\beta}\alpha b_i r_j + p_{3,k-1,i,1}\lambda\bar{\beta}\alpha r_j \\ &\quad + p_{3,k-1,i,j+1}\lambda\bar{\beta} + p_{3,k,i,1}\bar{\lambda}\bar{\beta}\alpha r_j + p_{3,k,i,j+1}\bar{\lambda}\bar{\beta}, \qquad k \ge 1,\ i \ge 1,\ j \ge 1, \end{aligned} \tag{6}$$

where $\bar{\beta} = 1-\beta,\ \bar{\alpha} = 1-\alpha,\ \bar{\lambda} = 1-\lambda.$

We define

$$G_B(x,z) = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} z^k x^i p_{1,k,i},\quad \varphi_B(z) = \sum_{k=1}^{\infty} z^k p_{1,k,1},\quad G_D(x) = \sum_{i=1}^{\infty} x^i p_{2,i},$$

$$G_R(x,y,z) = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} z^k x^i y^j p_{3,k,i,j},\quad \varphi_R(x,z) = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} z^k x^i p_{3,k,i,1}.$$

Multiplying (2) and (3) by x and x^i, respectively, summing over i, then multiplying by z and z^k, respectively, and then summing over k, it yields

$$\frac{x-\bar{\beta}\bar{\alpha}\sigma}{x} G_B(x,z) = \frac{\bar{\beta}\bar{\alpha}\sigma\big(B(x)-z\big)}{z}\varphi_B(z) + \bar{\beta}\bar{\alpha}\sigma\varphi_R(x,z) + \bar{\alpha}B(x)\big(\lambda z p_{0,0} - \bar{\lambda}\bar{\beta}p_{1,1,1}\big), \tag{7}$$

where $\sigma = \bar{\lambda} + \lambda z$.

Similarly, multiplying (4) by x^i and summing over i, it yields

$$\frac{x-1}{x} G_D(x) = -p_{2,1} + \beta D(x)\left[\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\pi_{\ell,m} + \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\widehat{\pi}_{\ell,m,n}\right]. \tag{8}$$

Finally, multiplying (5) and (6) by y and y^j, respectively, summing over j, multiply by x and x^i, respectively, summing over j, multiplying by z and z^k, respectively, and then summing over k, it yields

$$\begin{aligned} \frac{y-\bar{\beta}\sigma}{y} G_R(x,y,z) &= \frac{\bar{\beta}\alpha\sigma R(y)}{x} G_B(x,z) + \frac{\bar{\beta}\alpha\sigma\big(B(x)-z\big)R(y)}{z}\varphi_B(z) \\ &\quad + \bar{\beta}\sigma\big(\alpha R(y)-1\big)\varphi_R(x,z) + \alpha B(x)R(y)\big(\lambda z p_{0,0} - \bar{\lambda}\bar{\beta}p_{1,1,1}\big). \end{aligned} \tag{9}$$

We set $y = \bar{\beta}\sigma$ and substitute it into (9), it yields

$$\frac{\bar{\beta}\sigma-\bar{\beta}\sigma}{\bar{\beta}\sigma}G_R(x,\bar{\beta}\sigma,z)=\frac{\bar{\beta}\alpha\sigma R(\bar{\beta}\sigma)}{x}G_B(x,z)+\frac{\bar{\beta}\alpha\sigma\left(B(x)-z\right)R(\bar{\beta}\sigma)}{z}\varphi_B(z)$$
$$+\bar{\beta}\sigma\left(\alpha R(\bar{\beta}\sigma)-1\right)\varphi_R(x,z)+\alpha B(x)R(\bar{\beta}\sigma)\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right).$$

Hence, we can get $\varphi_R(x,z)$ as following

$$\varphi_R(x,z)=\frac{\alpha R(\bar{\beta}\sigma)}{\left(1-\alpha R(\bar{\beta}\sigma)\right)}\left[\frac{G_B(x,z)}{x}+\frac{\left(B(x)-z\right)}{z}\varphi_B(z)+\frac{B(x)}{\bar{\beta}\sigma}\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right)\right]. \quad (10)$$

Substituting (10) into (7), we obtain

$$\left(1-\frac{\bar{\beta}\bar{\alpha}\sigma}{x\left(1-\alpha R(\bar{\beta}\sigma)\right)}\right)G_B(x,z)=\frac{\bar{\beta}\bar{\alpha}\sigma\left(B(x)-z\right)}{z\left(1-\alpha R(\bar{\beta}\sigma)\right)}\varphi_B(z)$$
$$+\frac{\bar{\alpha}B(x)}{\left(1-\alpha R(\bar{\beta}\sigma)\right)}\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right). \quad (11)$$

We set $x=\eta=\dfrac{\bar{\beta}\bar{\alpha}\sigma}{1-\alpha R\left(\bar{\beta}\sigma\right)}=\dfrac{\bar{\beta}\bar{\alpha}(\bar{\lambda}+\lambda z)}{1-\alpha R\left(\bar{\beta}(\bar{\lambda}+\lambda z)\right)}$ and substitute it into (11), it yields

$$\left[1-\frac{\bar{\beta}\bar{\alpha}\sigma}{\eta\left(1-\alpha R(\bar{\beta}\sigma)\right)}\right]G_B(\eta,z)=\frac{\bar{\beta}\bar{\alpha}\sigma\left(B(\eta)-z\right)}{z\left(1-\alpha R(\bar{\beta}\sigma)\right)}\varphi_B(z)+\frac{\bar{\alpha}B(\eta)}{\left(1-\alpha R(\bar{\beta}\sigma)\right)}\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right)$$
$$0=\frac{\bar{\beta}\bar{\alpha}\sigma\left(B(\eta)-z\right)}{z\left(1-\alpha R(\bar{\beta}\sigma)\right)}\varphi_B(z)+\frac{\bar{\alpha}B(\eta)}{\left(1-\alpha R(\bar{\beta}\sigma)\right)}\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right).$$

Then we can obtain

$$\varphi_B(z)=\frac{zB(\eta)}{\bar{\beta}\sigma\left(z-B(\eta)\right)}\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right). \quad (12)$$

Substituting $\phi_B(z)$ into (11), it yields

$$\left[1-\frac{\bar{\beta}\bar{\alpha}\sigma}{x\left(1-\alpha R(\bar{\beta}\sigma)\right)}\right]G_B(x,z)=\frac{\bar{\beta}\bar{\alpha}\sigma\left(B(x)-z\right)}{z\left(1-\alpha R(\bar{\beta}\sigma)\right)}\left[\frac{zB(\eta)}{\bar{\beta}\sigma\left(z-B(\eta)\right)}\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right)\right]$$
$$+\frac{\bar{\alpha}B(x)}{\left(1-\alpha R(\bar{\beta}\sigma)\right)}\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right).$$

Then we can obtain

$$G_B(x,z)=\frac{xz\bar{\alpha}\left(B(x)-B(\eta)\right)}{\left(z-B(\eta)\right)\left[x\left(1-\alpha R(\bar{\beta}\sigma)\right)-\bar{\beta}\bar{\alpha}\sigma\right]}\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right). \quad (13)$$

Finally, we replace the term $\varphi_R(x,z)$ of (9) by (10), it yields

$$\varphi_R(x,z)=\frac{\alpha R(\bar{\beta}\sigma)}{\left(1-\alpha R(\bar{\beta}\sigma)\right)}\left[\frac{G_B(x,z)}{x}+\frac{\left(B(x)-z\right)}{z}\varphi_B(z)+\frac{B(x)}{\bar{\beta}\sigma}\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right)\right],$$

$$\begin{aligned}\frac{y-\bar{\beta}\sigma}{y}G_R(x,y,z)&=\frac{\bar{\beta}\alpha\sigma R(y)}{x}G_B(x,z)+\frac{\bar{\beta}\alpha\sigma\left(B(x)-z\right)R(y)}{z}\varphi_B(z)\\&+\bar{\beta}\sigma\left(\alpha R(y)-1\right)\left\{\frac{\alpha R(\bar{\beta}\sigma)}{\left(1-\alpha R(\bar{\beta}\sigma)\right)}\left[\frac{G_B(x,z)}{x}+\frac{\left(B(x)-z\right)}{z}\varphi_B(z)+\frac{B(x)}{\bar{\beta}\sigma}\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right)\right]\right\}\\&+\alpha B(x)R(y)\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right),\end{aligned}$$

$$\frac{y-\bar{\beta}\sigma}{y}G_R(x,y,z)=\frac{\left(R(y)-R(\bar{\beta}\sigma)\right)}{\left(1-\alpha R(\bar{\beta}\sigma)\right)}\left[\frac{\alpha\bar{\beta}\sigma G_B(x,z)}{x}+\frac{\alpha\bar{\beta}\sigma\left(B(x)-z\right)}{z}\varphi_B(z)+\alpha\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right)B(x)\right].$$

The terms $G_B(x,z)$ and $\varphi_B(z)$ of above equations can be replaced by (13) and (12), respectively. It yields

$$\frac{y-\bar{\beta}\sigma}{y}G_R(x,y,z)=\frac{\left(R(y)-R(\bar{\beta}\sigma)\right)}{\left(1-\alpha R(\bar{\beta}\sigma)\right)}\left[\begin{array}{l}\dfrac{\alpha\bar{\beta}\sigma\dfrac{xz\bar{\alpha}\left(B(x)-B(\eta)\right)}{\left(z-B(\eta)\right)\left[x\left(1-\alpha R(\bar{\beta}\sigma)\right)-\bar{\beta}\bar{\alpha}\sigma\right]}\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right)}{x}\\+\dfrac{\alpha\bar{\beta}\sigma\left(B(x)-z\right)}{z}\left(\dfrac{zB(\eta)}{\bar{\beta}\sigma\left(z-B(\eta)\right)}\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right)\right)+\alpha\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right)B(x)\end{array}\right].$$

Then we have the following equations:

$$G_R(x,y,z)=\frac{xyz\alpha\left(B(x)-B(\eta)\right)\left(R(y)-R(\bar{\beta}\sigma)\right)}{\left(y-\bar{\beta}\sigma\right)\left(z-B(\eta)\right)\left(x-x\alpha R(\bar{\beta}\sigma)-\bar{\beta}\bar{\alpha}\sigma\right)}\left(z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}\right). \qquad (14)$$

In (12), $p_{0,0}$ and $p_{1,1,1}$ are unknown. Using Rouche's theorem, the denominator $z-B(\eta)$ has a unique solution within a unit circle $|z|<1$. Let $\hat{z}$ be the unique root of $z-B(\eta)=0$. The numerator $z\lambda p_{0,0}-\bar{\lambda}\bar{\beta}p_{1,1,1}$ should be zero under $z=\hat{z}$. We obtain

$$p_{1,1,1} = \frac{\lambda \hat{z}}{\bar{\lambda}\bar{\beta}} p_{0,0} . \tag{15}$$

Setting $x = 1$ in (8), we obtain

$$\left[\sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \pi_{\ell,m} + \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \hat{\pi}_{\ell,m,n} \right] = \frac{p_{2,1}}{\beta}, \tag{16}$$

$$G_D(x) = \frac{x\left(D(x)-1\right)}{x-1} p_{2,1} . \tag{17}$$

From (17), we have

$$G_D(1) = D^{(1)} p_{2,1} . \tag{18}$$

From (13)-(15), and the equality $G_B(1,1) + G_R(1,1,1) = \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} p_{1,\ell,m} + \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{3,\ell,m,n}$, we get

$$p_{2,1} = \lambda\left(1 - \hat{z}\right) p_{0,0} . \tag{19}$$

Using the normalizing condition $p_{0,0} + G_B(1,1) + G_R(1,1,1) + G_D(1) = 1$, we obtain

$$p_{0,0} = \frac{\beta}{\beta + \lambda\left(1 + \beta D^{(1)}\right)\left(1 - \hat{z}\right)} . \tag{20}$$

Theorem 1. *The stationary distribution of the Markov chain* $Y = \{(\Gamma^{(n)}, L^{(n)}, \varepsilon^{(n)}, \xi^{(n)}), n = 0,1,2,...\}$ *has the following generating functions:*

$$G_B(x,z) = \frac{xz\lambda\bar{\alpha}\left(B(x) - B(\eta)\right)\left(z - \hat{z}\right)}{\left(z - B(\eta)\right)\left[x\left(1 - \alpha R(\bar{\beta}\sigma)\right) - \bar{\beta}\bar{\alpha}\sigma\right]} p_{0,0} \ ;$$

$$G_R(x,y,z) = \frac{xyz\lambda\alpha\left(B(x) - B(\eta)\right)\left(R(y) - R(\bar{\beta}\sigma)\right)\left(z - \hat{z}\right)}{\left(y - \bar{\beta}\sigma\right)\left(z - B(\eta)\right)\left(x - x\alpha R(\bar{\beta}\sigma) - \bar{\beta}\bar{\alpha}\sigma\right)} p_{0,0} \ ;$$

$$G_D(x) = \frac{x\left(D(x)-1\right)}{x-1} p_{2,1} \ ,$$

where $\eta = \dfrac{\bar{\beta}\bar{\alpha}\sigma}{1 - \alpha R(\bar{\beta}\sigma)}$ and $p_{0,0} = \dfrac{\beta}{\beta + \lambda\left(1 + \beta D^{(1)}\right)\left(1 - \hat{z}\right)}$.

Corollary 1.

(i) *When the server is busy, the generating function of the system size is given as follows:*

$$G_B(1,z)=\frac{z\lambda\bar{\alpha}\beta\left(1-B(\eta)\right)\left(z-\hat{z}\right)}{\left(z-B(\eta)\right)\left(1-\alpha R(\bar{\beta}\sigma)-\bar{\beta}\bar{\alpha}\sigma\right)\left(\beta+\lambda\left(1+\beta D^{(1)}\right)\left(1-\hat{z}\right)\right)}.$$

(ii) *When the server is under repair of a non-disastrous failure, the generating function of the system size is given as follows:*

$$G_R(1,1,z)=\frac{z\lambda\alpha\beta\left(1-B(\eta)\right)\left(1-R(\bar{\beta}\sigma)\right)\left(z-\hat{z}\right)}{\left(1-\bar{\beta}\sigma\right)\left(z-B(\eta)\right)\left(1-\alpha R(\bar{\beta}\sigma)-\bar{\beta}\bar{\alpha}\sigma\right)\left(\beta+\lambda\left(1+\beta D^{(1)}\right)\left(1-\hat{z}\right)\right)}.$$

Corollary 2. *The state probabilities of the server in the steady-state are given as follows:*

$$P_I=P\left(\Gamma=0\right)=\frac{\beta}{\beta+\lambda\left(1+\beta D^{(1)}\right)\left(1-\hat{z}\right)},$$

$$P_B=P\left(\Gamma=1\right)=G_B(1,1)=\frac{\lambda\bar{\alpha}\beta\left(1-\hat{z}\right)}{\left(1-\alpha R(\bar{\beta})-\bar{\beta}\bar{\alpha}\right)\left(\beta+\lambda\left(1+\beta D^{(1)}\right)\left(1-\hat{z}\right)\right)},$$

$$P_D=P\left(\Gamma=2\right)=G_D(1)=\frac{\lambda\beta D^{(1)}\left(1-\hat{z}\right)}{\beta+\lambda\left(1+\beta D^{(1)}\right)\left(1-\hat{z}\right)},$$

$$P_R=P\left(\Gamma=3\right)=G_R(1,1,1)=\frac{\lambda\alpha\left(1-R(\bar{\beta})\right)\left(1-\hat{z}\right)}{\left(1-\alpha R(\bar{\beta})-\bar{\beta}\bar{\alpha}\right)\left(\beta+\lambda\left(1+\beta D^{(1)}\right)\left(1-\hat{z}\right)\right)},$$

$$P_{GB}=P_B+P_R=\frac{\lambda\left(1-\hat{z}\right)}{\beta+\lambda\left(1+\beta D^{(1)}\right)\left(1-\hat{z}\right)}.$$

4. System Performance Measures

In this section, we derive the system performance measures such as the busy cycle, the idle period, the generalized busy period, and the disastrous period. The system size and the sojourn time are also derived.

4.1. Busy cycle, idle period, generalized busy period, and disastrous period

The periods of the server states are divided into the idle period (IP), the generalized busy period (GBP), and the disastrous period (DP). An IP begins at the server becoming free and terminates at a costumer arrival. A GBP (which consists of a busy period and a possible non-disastrous period) starts at the beginning of a generalized service and ends at that the generalized service has been completed and simultaneously the system is empty, or a disastrous failure occurs. The generalized service begins at a customer starting its service and terminates at the service being completed. The generalized service time includes the repair time of a non-disastrous failure. A DP starts at a disastrous failure occurring and ends at its repair being completed. A regeneration busy cycle is the interval which consists of an IP, a GBP and a DP. Let Φ be the generalized busy period and the PGF of Φ is given by

$$\Phi(z) = B\left(\frac{\bar{\alpha}\left(\lambda z\Phi(z) + \bar{\lambda} z\right)}{1-\alpha R\left(\lambda z\Phi(z) + \bar{\lambda} z\right)}\right).$$

There are two types of sub-cycle depending on a GBP terminated by a generalized service completion (type-1 sub-cycle) or a disastrous failure occurring (type-2 sub-cycle). A type-1 sub-cycle consists of GBP and IP. A type-2 sub-cycle consists of GBP, IP, and DP. Let π_1 and π_2 denote the probabilities of type-1 sub-cycle and type 2 sub-cycle, respectively. Then we have

$$\begin{aligned}\pi_1 = P\left[\Phi < \Lambda\right] &= \sum_{k=1}^{\infty} P\left[\Phi = k, \Lambda > k\right] = \sum_{k=1}^{\infty} P\left[\Phi = k\right] P\left[\Lambda > k\right] \\ &= \sum_{k=1}^{\infty} P\left[\Phi = k\right] \sum_{j=k+1}^{\infty} P\left[\Lambda = j\right] = \Phi(\bar{\beta}),\end{aligned}$$

and $\pi_2 = 1 - \pi_1$, where Λ is a random variable geometrically distributed with parameter β.

Next, let C_i and $\overline{\Phi}_i$ denote the cycle length and the GBP of type-*i*, *i*=1, 2, respectively. Then we have the following generation functions:

$$C_1(z) = \overline{\Phi}_1(z) I(z),$$
$$C_2(z) = \overline{\Phi}_2(z) I(z) D(z),$$

where

$$\overline{\Phi}_1(z) = E\left[z^{\Phi} \mid \Phi < \Lambda\right] = \frac{\Phi(\bar{\beta} z)}{\pi_1},$$

$$\overline{\Phi}_2(z) = E\left[z^{\Lambda} \mid \Phi \geq \Lambda\right] = \frac{z\beta\left(1 - \Phi(\bar{\beta} z)\right)}{(1 - z\bar{\beta})\pi_2},$$

$$I(z) = \frac{\lambda z}{1 - \bar{\lambda} z},$$

$$D(z) = \sum_{i=1}^{\infty} z^i d_i .$$

Hence, we obtain the unconditional generating functions of the GBP, the IP, the DP, and the busy cycle as follows:

$$\overline{\Phi}(z) = \pi_1 \Phi_1\left(\bar{\beta} z\right) + \pi_2 \Phi_2\left(\bar{\beta} z\right),$$

$$\bar{I}(z) = \pi_1 I(z) + \pi_2 I(z),$$

$$\bar{D}(z) = \pi_2 D(z),$$

$$\bar{C}(z) = \pi_1 C_1(z) + \pi_2 C_2(z).$$

The above generating functions yield the following expectations:

$$E[\bar{\Phi}] = \frac{1 - \Phi(\bar{\beta})}{\beta}, \tag{21}$$

$$E[\bar{I}] = \frac{1}{\lambda}, \tag{22}$$

$$E[\bar{D}] = \pi_2 D^{(1)}, \tag{23}$$

$$E[\bar{C}] = \frac{\beta + \lambda\left(1 + \beta D^{(1)}\right)\left(1 - \Phi(\bar{\beta})\right)}{\lambda\beta}. \tag{24}$$

Remark:

1. Because $\hat{z} = B\left(\dfrac{\bar{\beta}\bar{\alpha}(\bar{\lambda} + \lambda\hat{z})}{1 - \alpha R(\bar{\beta}(\bar{\lambda} + \lambda\hat{z}))}\right)$ and $\Phi(z)\big|_{z=\bar{\beta}} = B\left(\dfrac{\bar{\alpha}\bar{\beta}\left(\bar{\lambda} + \lambda\Phi(\bar{\beta})\right)}{1 - \alpha R\left(\bar{\beta}(\bar{\lambda} + \lambda\Phi(\bar{\beta})\right)}\right)$, we have $\hat{z} = \Phi(\bar{\beta})$.
2. Using (21)-(24) and the renew reward theorem, we have

$$P_I = \frac{E(\bar{I})}{E(\bar{C})} = \frac{\beta}{\beta + \lambda\left(1 + \beta D^{(1)}\right)\left(1 - \Phi(\bar{\beta})\right)},$$

$$P_{GB} = \frac{E(\bar{\Phi})}{E(\bar{C})} = \frac{\lambda\left(1 - \Phi(\bar{\beta})\right)}{\beta + \lambda\left(1 + \beta D^{(1)}\right)\left(1 - \Phi(\bar{\beta})\right)},$$

$$P_D = \frac{E(\bar{D})}{E(\bar{C})} = \frac{\lambda\beta D^{(1)}\pi_2}{\beta + \lambda\left(1 + \beta D^{(1)}\right)\left(1 - \Phi(\bar{\beta})\right)}.$$

The results are the same as Corollary 2.

4.2. System size

Let $S(z)$ denote the PGF of the system size. We have

$$S(z)=p_{0,0}+G_B(1,z)+G_R(1,1,z)=\frac{z\lambda\beta\left(1-B(\eta)\right)\left(z-\hat{z}\right)}{\left(z-B(\eta)\right)\left(1-\bar{\beta}\sigma\right)\left(\beta+\lambda\left(1+\beta D^{(1)}\right)\left(1-\hat{z}\right)\right)},$$

which leads to the expected system size and is given by

$$E(S)=\frac{\lambda\left(2-\hat{z}+\dfrac{\lambda\bar{\beta}\left(1-\hat{z}\right)}{\beta}-\dfrac{1-\hat{z}}{1-B\left(\dfrac{\bar{\alpha}\bar{\beta}}{1-\alpha R(\bar{\beta})}\right)}\right)}{\beta+\lambda\left(1+\beta D^{(1)}\right)\left(1-\hat{z}\right)}. \tag{25}$$

That is, $E(S)=\dfrac{\lambda p_{0,0}}{\beta}\left(2-\hat{z}+\dfrac{\lambda\bar{\beta}\left(1-\hat{z}\right)}{\beta}-\dfrac{\left(1-\hat{z}\right)}{\left(1-B(\eta)\right)}\right)$.

4.3. Sojourn time

In this section, we derive the PGF of the expected sojourn time in the system. A customer's sojourn time in the system is the time from entering the system to leaving the system (i.e. the sum of waiting time in the queue plus his service time). Firstly, setting $\alpha=0$ in $G_B(x,z)$ of Theorem 1, it yields the joint PGF of the system size of server busy state and the remaining service time excluding non-disastrous failures as follows.

$$G_{BO}(x,z)=\frac{xz\lambda\left(B(x)-B(\bar{\beta}\sigma)\right)\left(z-\hat{z}\right)}{\left(z-B(\bar{\beta}\sigma)\right)\left(x-\bar{\beta}\sigma\right)}p_{0,0}\,. \tag{26}$$

Replacing the service time B by the generalized service time H in (26), it yields

$$G_{GB}(x,z)=\frac{xz\lambda\left(H(x)-H(\bar{\beta}\sigma)\right)\left(z-\hat{z}\right)}{\left(z-H(\bar{\beta}\sigma)\right)\left(x-\bar{\beta}\sigma\right)}p_{0,0}\,. \tag{27}$$

To investigate the expected sojourn time of a customer, we consider the status of customer arrivals and the status of customers already in the system. We assume a tagged customer arrives at slot n and a disastrous failure does not occur during their generalized service time for customers already in the system. At slot n, a disastrous failure does not occur as the server is idle. Hence the spending time of an arriving customer in the system is its generalized service time. When a disastrous failure occurs (with probability β) or the server is under repair due to a disastrous failure, the arrival customer's sojourn time is zero. If a disastrous failure does not occur (with probability $\bar{\beta}$), the server is during generalized busy, and k customers in the queue stay in the system, this customer's sojourn time consists of his generalized service time, the remaining generalized service time of the customer being served, and the generalized service time of the preceding *k-1* customers in the queue.

Let $U(z)$ denote the PGF of the unfinished work at the arrival slot of a tagged customer for the present model. By Bernoulli arrivals see time averages and (27), $U(z)$ is given by

$$U(z)=\frac{1}{p_{0,0}+\bar{\beta}P_{GB}}\left(p_{0,0}+\bar{\beta}\frac{G_{GB}(z,H(z))}{zH(z)}\right)H(z)=\left[1+\frac{\bar{\beta}\lambda\left(H(z)-\bar{z}\right)}{z-\bar{\beta}(\bar{\lambda}+\lambda H(z))}\right]H(\bar{\beta})p_{0,0}$$
$$=\frac{p_{0,0}}{1-P_D}\left(\frac{\beta(\bar{\lambda}+\lambda H(z))+\lambda\bar{\bar{z}}-\bar{z}}{z-\bar{\beta}(\bar{\lambda}+\lambda H(z))}\right)H(z).$$

Let W be the sojourn time of the tagged customer. We can get the PMF and the PGF of the tagged customer's sojourn time as follows:

$$P(W=k)=P\left(U=k,\Lambda\geq k+1\right)+P\left(U\geq k,\Lambda=k\right)=\bar{\beta}^{k}P\left(U=k\right)+\bar{\beta}^{k-1}\beta P\left(U\geq k\right),k\geq 1;$$

$$W(z)=\frac{\beta z+\left(1-z\right)U(\bar{\beta}z)}{1-\bar{\beta}z}.$$

Corollary 3. *The expected sojourn time is given by*

$$E(W)=W'(z)=\frac{p_{0,0}}{\beta\left(p_{0,0}+\bar{\beta}P_{GB}\right)}\left[\frac{\beta+\lambda\bar{\beta}\left(1-\bar{z}\right)}{\beta}-\frac{\left(1-\bar{z}\right)H(\bar{\beta})}{(1-H(\bar{\beta}))}\right].$$

Remark: From Corollary 3 and (25), the Little's formula $E(S)=\lambda_{eff}E(W)$ can be confirmed, where $\lambda_{eff}=\lambda\left(p_{0,0}+\bar{\beta}P_{GB}\right)$.

5. Numerical Examples

In this section, we perform several numerical examples to study the influences of parameters on the expected sojourn time $E(W)$. In each example, service time, the repair time of a disastrous failure, and the repair time of a non-disastrous failure are investigated. In each experiment, three different distributions, geometric distribution (Geo), binomial distribution (B), and negative binomial distribution (NB) are performed. In first example, we study the effect of the non-disastrous failure rate α on $E(W)$. We also set $\lambda = 0.4$ and $\beta = 0.05$. The numerical results are shown in Figures 1-3. From the figures we can see that the $E(W)$ increases as α increases, for service time, the repair time of a disastrous failure and the repair time of a non-disastrous failure. It is reasonable because the average sojourn time of a customer in the computer network center increases as the breakdown rate of computers increases. Besides, the numerical results revealed that different distributions do not affect the effect of the non-disastrous failure rate α on $E(W)$.

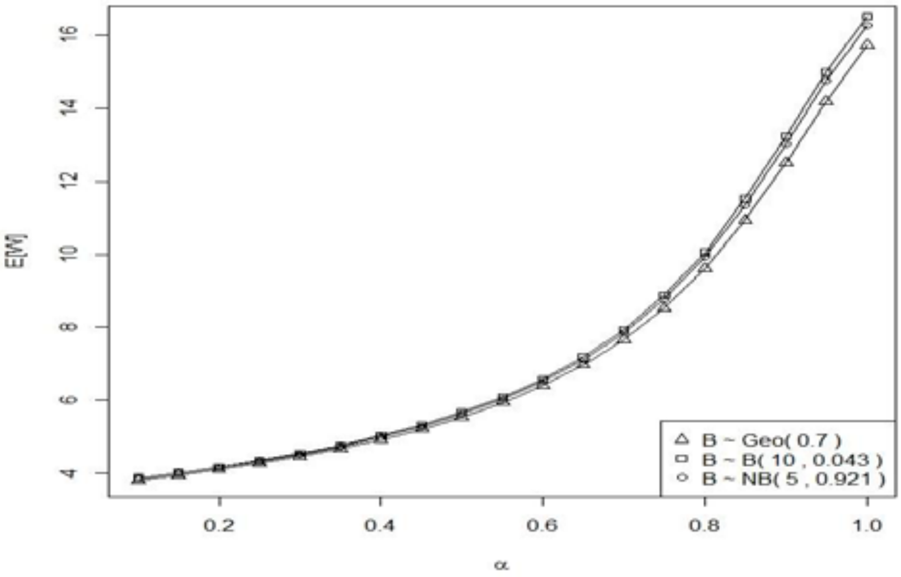

Figure 1. Effect of the service time distributions and α on $E(W)$.

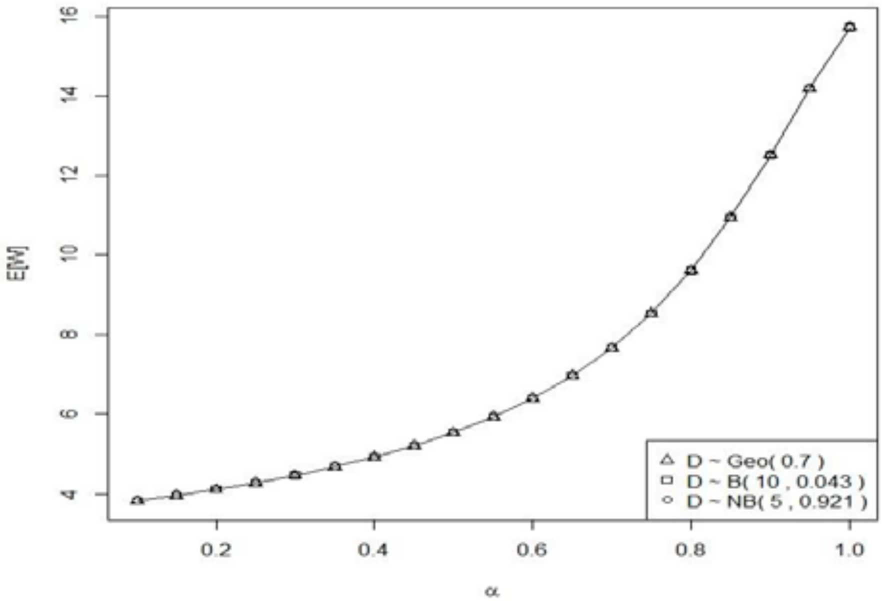

Figure 2. Effect of the repair time distributions of a disastrous failure and α on $E(W)$.

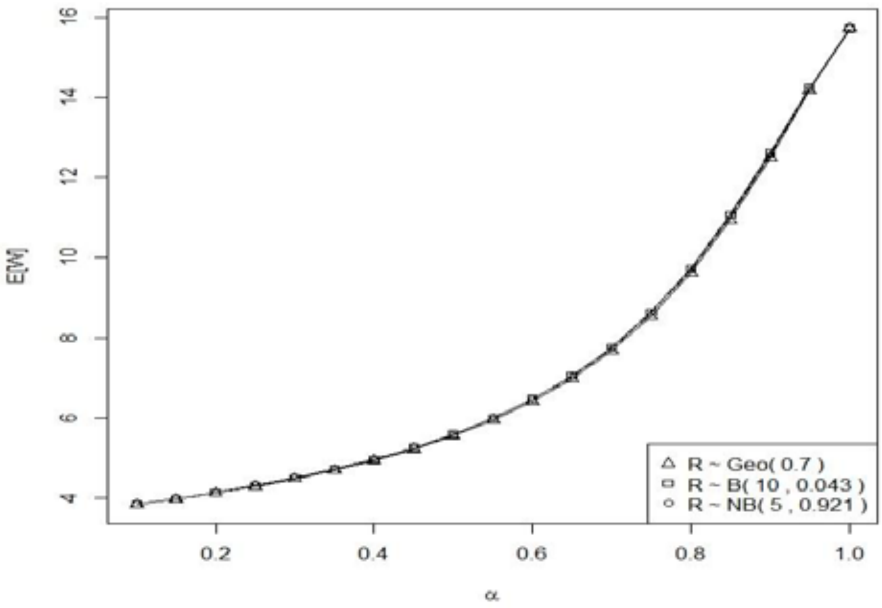

Figure 3. Effect of the repair time distributions of a non-disastrous failure and α on $E(W)$.

In second example, we study the impact of the disastrous failure rate β on $E(W)$. We also set $\lambda = 0.4$ and $\alpha = 0.03$. The numerical results are shown in Figures 4-6. The numerical results revealed that $E(W)$ decreases as β increases. This is because once the computer network center is affected by the virus, the computer network center removes all workloads and the customers are not allowed to enter into the system. In addition, the numerical results showed that different distributions do not affect the impact of the disastrous failure rate β on $E(W)$.

In third example, we study the effect of the arrival rate λ on $E(W)$. We also set $\beta = 0.05$ and $\alpha = 0.03$. The numerical results are shown in Figures 7-9. From the figures we can see that $E(W)$ increases as λ increases. It is reasonable because the average sojourn time of a customer in the computer network center increases as the arrival rate of customer increases. The numerical results also revealed that different distributions do not affect the effect of the arrival rate λ on $E(W)$.

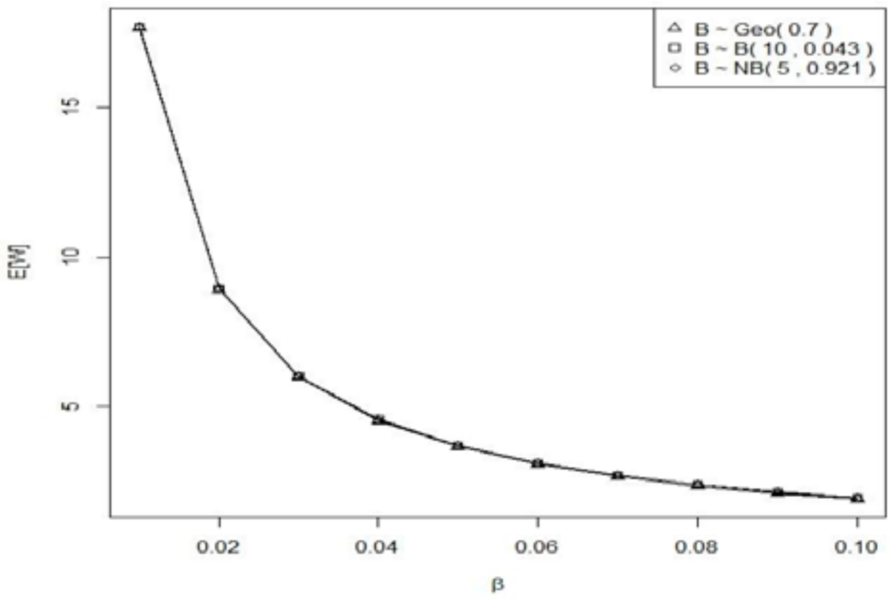

Figure 4. Effect of the service time distributions and β on $E(W)$.

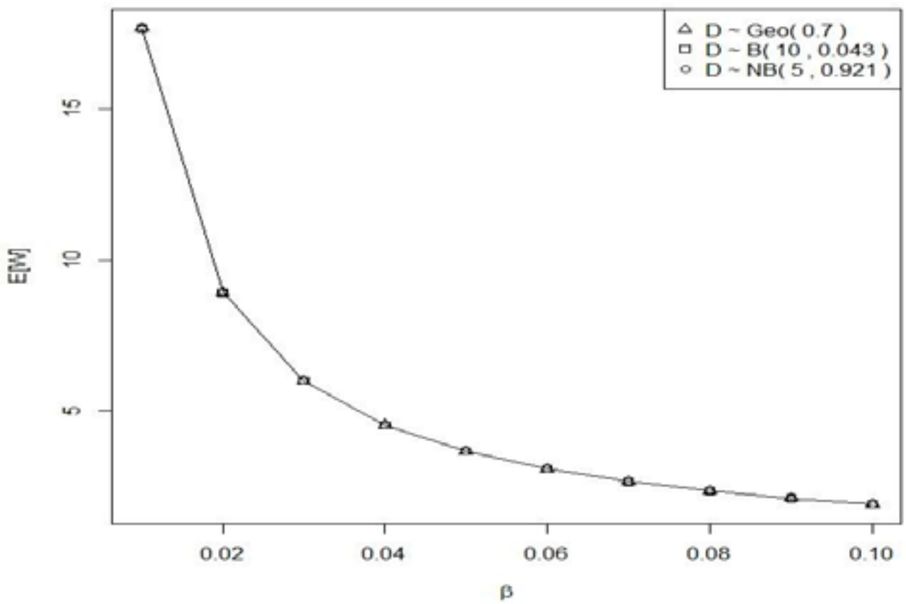

Figure 5. Effect of the repair time distributions of a disastrous failure and β on $E(W)$.

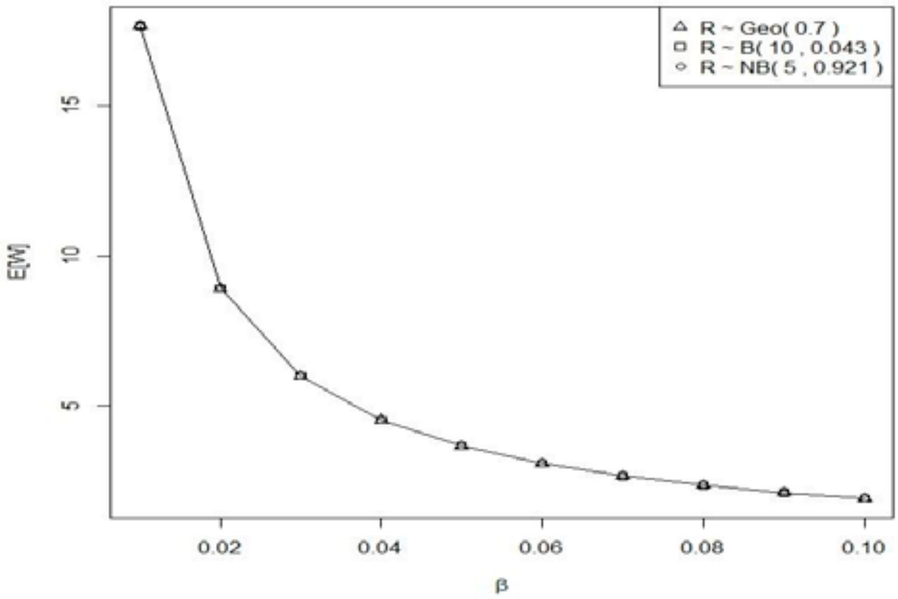

Figure 6. Effect of the repair time distributions of a non-disastrous failure and β on $E(W)$.

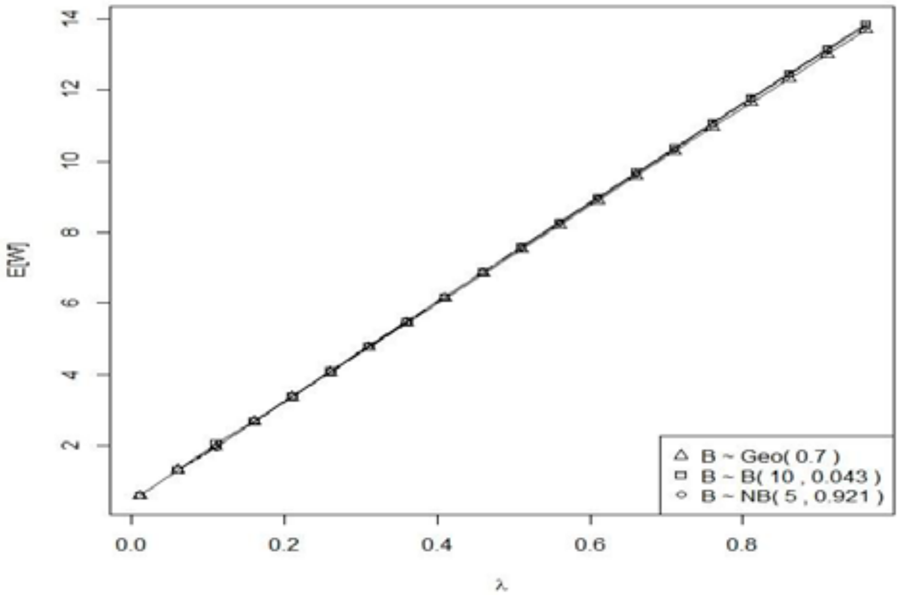

Figure 7. Effect of the service time distributions and λ on $E(W)$.

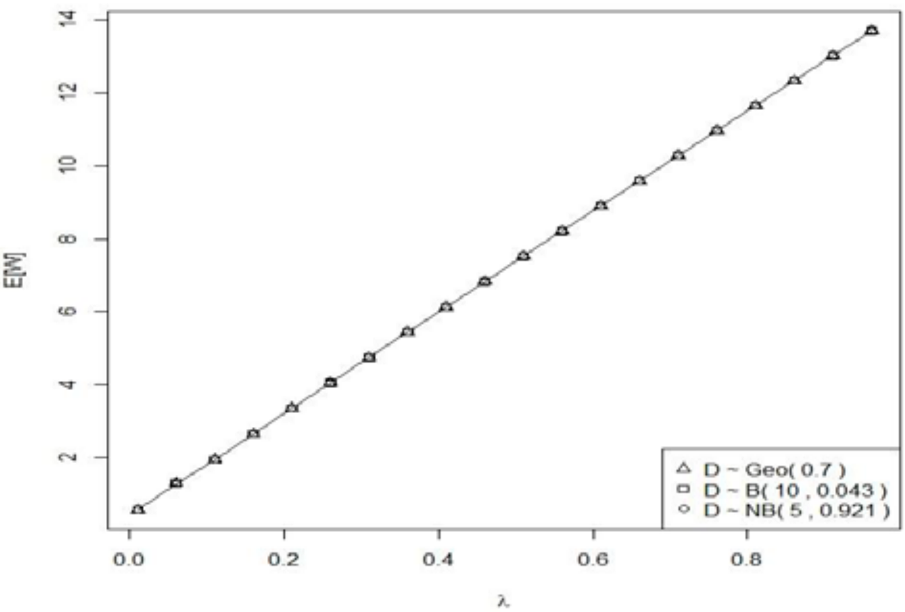

Figure 8. Effect of the repair time distributions of a disastrous failure and λ on $E(W)$.

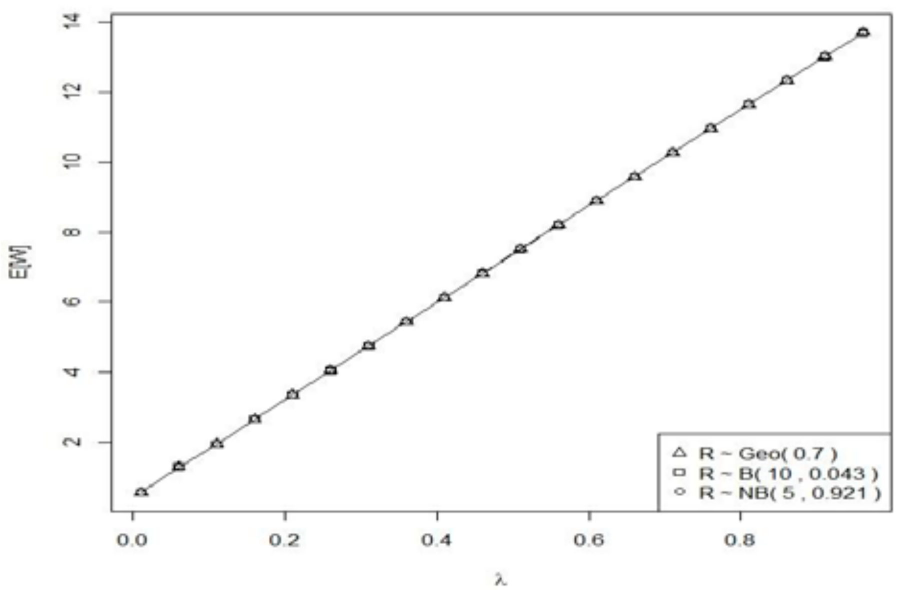

Figure 9. Effect of the repair time distributions of a non-disastrous failure and λ on $E(W)$.

6. Conclusions

This paper discussed a Geo/G/1 queue system with disastrous and non-disastrous failures. Different with other approaches, the investigated system may also suffer from a non-disastrous failure, which is caused by normal breakdown. During the repair process of a non-disastrous failure, arriving customers are allowed to join the system and in the meanwhile a disastrous failure may also occur. We have derived the stationary system size and the sojourn time distributions by using supplementary variable technique. We also derived some system characteristics such as the idle period and the busy cycle. Finally, we provided some numerical examples and numerical results to demonstrate the parameter effects on the expected sojourn time.

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