Queueing Models and
Service Management
Vol. 9, No. 3, page 57–83, 2026

QMSM

A Queueing Model with Two Types of MAPs Service During the Positive Inventory Using (s, S) Inventory Policy Management Method, Vacation, Setup, and Balking

V. Ganesan[1], Aliakbar Montazer Haghighi[2,*] and G. Ayyappan[3]

[1]Department of Mathematics
Sri Venkateshwaraa College of Engineering and Technology, Puducherry 605 102, India

[2]Department of Mathematics
Prairie View A&M University, Prairie View, Texas 77446, USA

[3]Department of Mathematics,
Puducherry Technological University, Puducherry 605 014, India

(*Received September 2025; accepted August 2026*)

Abstract: In this paper we consider a retrial inventory queueing model with two types of arrivals according to a Markovian arrival process (MAP), with a single-server retrial inventory, phase-type service during the positive inventory in all forms of services, balking, close down, single vacation, and use of the (s, S) inventory policy management method, where the reorder point (s) is set below the desired stock level (S) to ensure that inventory is replenished as needed. An arriving item when the server is busy will join an infinite capacity orbit or balks. The server closes the serving process, and goes on a limited vacation time, if there is no item in orbit and it is being idle. Upon return from vacation, the server will proceed with a set up period before proceeding to serve if there is any item waiting. Accordingly, we will find the steady-state probability vector, the busy period, performance measures, cost analysis, and finally, some numerical examples.

Keywords: Balking, close down, Markovian arrival process, matrix analytic method, phase type distribution, retrial inventory queue, (s, S) replenishment policy, setup, single vacation, two way communication.

1. Introduction

Recent progress in computer networks and communication systems has significantly increased research interest in inventory retrial queueing models because of their wide range of practical applications. In a queueing-inventory system, a customer receives an item from the inventory after the completion of service. A major contribution to this area was made by [18], who introduced the Markovian Arrival Process (MAP), a flexible class of point processes

* Corresponding author
Email: amhaghighi@pvamu.edu

closely associated with finite-state Markov chains. Several well-known arrival processes can be viewed as special cases of MAP, including Markov-modulated Poisson processes, phase-type renewal processes, and certain semi-Markov point processes. Later, [9] provided a detailed exposition of MAPs and phase-type distributions, highlighting their ability to represent both correlated and independent arrival patterns. Furthermore, [20] developed matrix-analytic methods that became fundamental tools for analyzing complex queueing systems. In another study, [13] examined a single-server queueing model with Poisson arrivals, incorporating three buffers, a splitter, and a feedback mechanism. The feedback process was characterized by bounded random batch sizes and exponential delays, providing additional insight into the behavior of such systems.

The field of retrial queueing theory has been significantly enriched by several important contributions. [11] presented the first comprehensive monograph on retrial queues, offering an in-depth discussion of the underlying methodologies, analytical techniques, and key theoretical results. The work covers a wide range of retrial queueing models, including both single-server and multi-server systems. To facilitate further research in this area, [3] compiled an extensive bibliography that serves as a valuable reference source for publications related to retrial queues. Moreover, [23] provided a thorough survey of retrial queueing models and their applications. [1] investigated an M/M/1 queueing–inventory system with batch demand and limited waiting space, derives steady-state probabilities, optimizes inventory parameters, evaluates system performance, and validates analytical results through simulation experiments. [21] analyzed an M/G/1 queue with optional second service, applies a modified Gamma kernel estimation method, and evaluates performance measures through simulation studies. The survey reviewed models arising from practical situations, summarized major research developments, and identified several open problems and promising directions for future investigation.

The development of queueing–inventory systems has attracted considerable attention in recent years. [4] analyzed a perishable inventory model involving two different commodities and optional customer demands. The study obtained steady-state probabilities using matrix-geometric techniques and assessed various performance characteristics through numerical experiments. Further extending this area, [10] examined a single-server queueing–inventory system with MAP arrivals, phase-type service distributions, customer feedback, product returns, a finite retrial orbit, balking behavior, and an (s, S) inventory policy, with the objective of improving system efficiency and profitability. Earlier, [25] investigated an inventory system under the assumption that customer demands follow a renewal process and that replenishment orders are readily available. The model incorporated a service facility together with an unlimited waiting space, demonstrating the close relationship between inventory management and queueing theory in analyzing service-oriented systems.

Server failures and service interruptions are important factors in practical queueing systems because they can considerably affect system efficiency. [15] examined an inventory retrial queueing model with server interruptions and incorporated an (s, S) replenishment policy. In this model, customer arrivals are described by a Poisson process, while both service durations and replenishment lead times are assumed to follow exponential distributions.

[24] analyzed a retrial inventory queueing system by explicitly considering the occurrence of server breakdowns. [6] investigated a two-commodity inventory queueing model that incorporates multiple vacation periods, working breakdowns, and emergency replenishment. Similarly, [14] studied the busy period of a single-server queue with delayed feedback, where returning customers are generated in batches according to an exponential delay mechanism. Furthermore, [7] considered a preemptive-priority queueing system involving impatient customers, a single vacation, and server repair.

[2] investigated the steady-state behaviour of an $M/G/1$ retrial queue with two-way communication operating through a single channel. [8] examined a queueing system in which a standby server provides service when the primary server fails, incorporating a constant retrial rate and collisions among orbiting customers. [17] studied a single-server Markovian retrial queue with a finite source population and two-way communication. In their model, exponential distributions were employed for both incoming-call service and outgoing-call service times. [5] considered a single-server constant-retrial queueing model incorporating several operational features, including shutdown, vacation, optional service, setup, and customer balking. Furthermore, [12] developed a two-station single-server tandem queueing model that incorporated work splitting, feedback, and blocking mechanisms.

The salient features of the proposed model are as follows:

- A comprehensive single-server retrial queueing inventory system is developed by integrating MAP arrivals, inventory control and two-way communication within a unified framework. This model considers a two-stage service mechanism consisting of essential and optional services, where customer behavior after essential service is governed by probabilistic decisions.
- The stochastic behavior of the system is formulated through matrix-analytic techniques, facilitating the computation of steady-state probabilities and performance measures. We derived stability condition of the model explicitly. We also discussed busy period of the system under consideration. Important performance indicators such as mean orbit size, average inventory levels, server-state probabilities and customer loss measures characteristics are obtained.
- Numerical examples are analysed to under stand the behaviour of the system. Total expected cost function is formulated by incorporating inventory holding costs, replenishment costs, setup costs, vacation costs, and customer-loss costs. A Numerical experiment is conducted to identify optimal inventory control parameters and minimize the overall system cost.

The proposed model provides practical guidelines for designing efficient service and inventory control policies in applications involving customer retrials, inventory-dependent services, and two way communication activities.

The organization of this paper is as follows. Section 2 introduces the proposed queueing–inventory model and presents its schematic framework. Section 3 develops the state-space structure and constructs the associated QBD process through the matrix-analytic approach. In Section 4, the stability condition is derived and the steady-state probability vectors are obtained. Section 5 is devoted to the analysis of the system's busy period. The key performance measures are formulated and examined in Section 6. Section 7 develops the expected total cost function and discusses its components. Numerical investigations, accompanied by two-dimensional and three-dimensional graphical illustrations, are presented in Section 8 to demonstrate the impact of various system parameters. Section 9 explores the optimal cost behavior under different operating conditions. Finally, Section 10 summarizes the main contributions of the study and outlines the concluding observations.

2. Model Description

This study examines a single-server retrial inventory queueing model in which incoming customers arrivals follow a Markovian Arrival Process (MAP) represented by matrices (D_0, D_1) of order m_1. Here, D_0 denotes no arrival and D_1 denotes an arrival to the system. An incoming call receives immediate service only if the server is idle and the inventory level is positive. Otherwise, the arriving customer either joins an infinite-capacity orbit with probability q_1 or balks with probability p_1. Upon service completion, if no inventory is available for optional service, the customer leaves the system. If inventory is available, the customer proceeds to optional service with probability p_2 or exits the system with probability q_2. Following completion of optional service, the customer exits the system. The server initiates essential service if an incoming customer or retrial customer is present. If the orbit contains customers, the server serves an orbital customer; otherwise, the server begins the close-down process and proceeds on vacation. After a single vacation, the server undergoes a setup process. Upon setup completion, the server provides service if an incoming or retrial customer is present; otherwise, the server remains idle. During idle periods with positive inventory, the server initiates outgoing calls, which follow a phase-type distribution represented by (β_1, T_1) of order n_1, (β_2, T_2) of order n_2, and (β_3, T_3) of order n_3 for incoming, optional, and outgoing call service times respectively. Balking of incoming arrivals occurs when the server is occupied with service, close-down, vacation, or setup processes, and when the inventory of essential items is depleted. Inter-retrial times for each customer in the infinite-capacity orbit follow an exponential distribution with constant rate ψ. The durations of close-down, vacation, and setup processes, denoted by ϕ, η and ξ, respectively, also follow exponential distribution. Inventory replenishment is governed by an (s, S) policy, with replenishment durations for essential and optional items following exponential distribution with parameters σ_1 and σ_2, respectively. Also the structure of the proposed system is shown in the Figure 1.

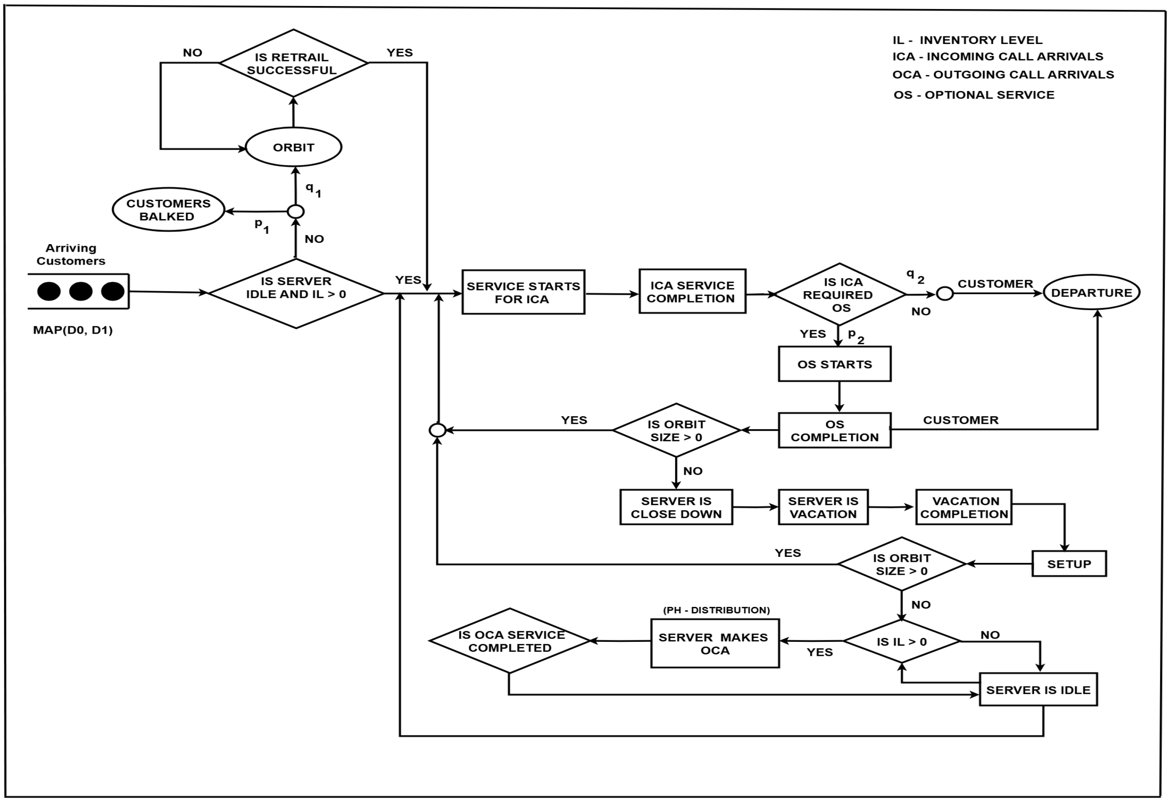

Figure 1. A schematic representation

3. The QBD Processes Under Infinitesimal Matrix Generations

We are going to discuss this part, which comprises the notation that forms the basis of the Quasi Birth and Death (QBD) process in our design.

- $\otimes -$ Any two matrices with different order are multiplied by using a Kronecker product, this part of work was done by the research of [22].
- $\oplus -$ Any two different order matrices can be taken for sum by using a Kronecker sum.
- $I_k -$ The identity matrix has k dimensions.
- $e -$ The column vector's each of its elements 1 is taken part of the appropriate dimension as we need.
- $e_k -$ The column vector for every k element is 1.
- Arrival rate of incoming customers is represented by λ and described as $\lambda = \pi D_1 e$.
- The essential service rate for incoming call arrivals represented by μ_1 and described as $\mu_1 = [\beta_1(-T_1)^{-1}e_{n_1}]^{-1}$.
- The optional service rate for incoming customer arrivals represented by μ_2 and described as $\mu_2 = [\beta_2(-T_2)^{-1}e_{n_2}]^{-1}$.
- The service rate for outgoing customer arrivals represented by μ_3 and described as $\mu_3 = [\beta_3(-T_3)^{-1}e_{n_3}]^{-1}$.
- Rate of that server makes the outgoing customer, where $\mu_4 = [\alpha(-U)^{-1}e_{m_2}]^{-1}$.
- $N(t)$ describes that number of consumers in the orbit of infinite capacity at time t.

- $Y(t)$ describes that the level of server status at time t, where

$$Y(t)=\begin{cases}0, & \text{server being an idle, if } I_1(t)=0;\\ & \text{makes outgoing calls, if } I_1(t)\geq 1,\\ 1, & \text{server is offering essential service for incoming customer arrivals,}\\ 2, & \text{server is offering optional service for incoming customer arrivals,}\\ 3, & \text{server is offering service for outgoing customer arrivals,}\\ 4, & \text{server is under close down process,}\\ 5, & \text{server is in vacation, and}\\ 6, & \text{server is under setup process.}\end{cases}$$

- $I_1(t)$ be the essential inventory items at time t.
- $I_2(t)$ be the optional inventory items at time t.
- $J_1(t)$ be the essential service for incoming customer arrivals by the server at time t.
- $J_2(t)$ be the optional service for incoming customer arrivals by the server at time t.
- $J_3(t)$ be the outgoing customer service by the server at time t.
- $M_1(t)$ be the incoming customer arrivals at time t.
- $M_2(t)$ be the outgoing customer arrivals at time t.

Let $\{N(t), Y(t), I_1(t), I_2(t), J_1(t), J_2(t), J_3(t), M_1(t), M_2(t) : t \geq 0\}$ represent a CTMC with state-level independent QBD processes whose state space is as follows:

$$\Omega=\bigcup_{i=0}^{\infty} l(i), \quad \text{where}$$

$$l(i)=\begin{cases}\{(i,0,0,j_2,k_1): 0\leq j_2\leq S_2, 1\leq k_1\leq m_1\},\\ \cup\{(i,0,j_1,j_2,k_1,k_2): 1\leq j_1\leq S_1, 0\leq j_2\leq S_2, 1\leq k_1\leq m_1, 1\leq k_2\leq m_2\},\\ \cup\{(i,1,j_1,j_2,r_1,k_1): 1\leq j_1\leq S_1, 0\leq j_2\leq S_2, 1\leq r_1\leq n_1, 1\leq k_1\leq m_1\},\\ \cup\{(i,2,j_1,j_2,r_2,k_1): 0\leq j_1\leq S_1, 1\leq j_2\leq S_2, 1\leq r_2\leq n_2, 1\leq k_1\leq m_1\},\\ \cup\{(i,3,j_1,j_2,r_3,k_1): 1\leq j_1\leq S_1, 0\leq j_2\leq S_2, 1\leq r_3\leq n_3, 1\leq k_1\leq m_1\},\\ \cup\{(i,u,j_1,j_2,k_1): u=4,5,6, 0\leq j_1\leq S_1, 0\leq j_2\leq S_2, 1\leq k_1\leq m_1\} : i\geq 0.\end{cases}$$

The infinitesimal matrix generator of the QBD process is written as follows:

$$Q=\begin{bmatrix}A_{00} & V_0 & 0 & 0 & 0 & \dots\\ V_2 & V_1 & V_0 & 0 & 0 & \dots\\ 0 & V_2 & V_1 & V_0 & 0 & \dots\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots\end{bmatrix}. \tag{1}$$

In the following, the ingress of the matrix Q structures are presented.

$$A_{00} = \begin{bmatrix} A_{00}^{11} & A_{00}^{12} & 0 & A_{00}^{14} & 0 & 0 & 0 \\ A_{00}^{21} & A_{00}^{22} & A_{00}^{23} & 0 & A_{00}^{25} & 0 & 0 \\ 0 & 0 & A_{00}^{33} & 0 & A_{00}^{35} & 0 & 0 \\ 0 & 0 & 0 & A_{00}^{44} & A_{00}^{45} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{00}^{55} & A_{00}^{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{00}^{66} & A_{00}^{67} \\ A_{00}^{71} & 0 & 0 & 0 & 0 & 0 & A_{00}^{77} \end{bmatrix}, \tag{2}$$

$$A_{00}^{11} = \begin{bmatrix} C_1 & 0 & \dots & 0 & 0 & \dots & C_2 \\ 0 & C_3 & \dots & 0 & 0 & \dots & C_4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & C_3 & 0 & \dots & C_4 \\ 0 & 0 & \dots & 0 & C_5 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & C_5 \end{bmatrix}. \ C_1 = \begin{bmatrix} J_1 & \dots & 0 & 0 & \dots & J_2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_1 & 0 & \dots & J_2 \\ 0 & \dots & 0 & J_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_3 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} J_4 & \dots & 0 & 0 & \dots & J_5 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_4 & 0 & \dots & J_5 \\ 0 & \dots & 0 & J_6 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_6 \end{bmatrix}, \ C_5 = \begin{bmatrix} J_7 & \dots & 0 & 0 & \dots & J_5 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_7 & 0 & \dots & J_5 \\ 0 & \dots & 0 & J_8 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_8 \end{bmatrix},$$

$$C_2 = \left[I_{S_2+1} \otimes \alpha \otimes \sigma_1 I_{m_1}, \right] \quad C_4 = \left[I_{S_2+1} \otimes \sigma_1 I_{m_2 m_1} \right],$$

$$J_1 = (D_0 + p_1 D_1) - (\sigma_1 + \sigma_2) I_{m_1}, \quad J_2 = e_{S_2+1} \otimes \sigma_2 I_{m_1}, \quad J_3 = (D_0 + p_1 D_1) - \sigma_1 I_{m_1},$$

$$J_4 = U \oplus (D_0 + p_1 D_1) - (\sigma_1 + \sigma_2) I_{m_2 m_1}, \quad J_6 = U \oplus (D_0 + p_1 D_1) - \sigma_1 I_{m_2 m_1},$$

$$J_5 = e_{S_2+1} \otimes \sigma_2 I_{m_2 m_1}, \quad J_7 = U \oplus (D_0 + p_1 D_1) - \sigma_2 I_{m_2}, \quad J_8 = U \oplus (D_0 + p_1 D_1),$$

$$A_{00}^{12} = \begin{bmatrix} 0 \\ I_{S_1} \otimes I_{S_2+1} \otimes e_{m_2} \otimes \beta_1 \otimes D_1 \end{bmatrix}, \ A_{00}^{14} = \begin{bmatrix} 0 \\ I_{S_1} \otimes I_{S_2+1} \otimes U^0 \otimes \beta_3 \otimes I_{m_1} \end{bmatrix},$$

$$A_{00}^{21} = \begin{bmatrix} C_6 & 0 & 0 \\ 0 & I_{S_1-1} \otimes C_7 & 0 \end{bmatrix}, \ C_6 = \begin{bmatrix} T_1^0 \otimes I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \ C_7 = \begin{bmatrix} T_1^0 \otimes \alpha \otimes I_{m_1} & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{00}^{22} = \begin{bmatrix} C_8 & \dots & 0 & 0 & \dots & C_9 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & C_8 & 0 & \dots & C_9 \\ 0 & \dots & 0 & C_{10} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & C_{10} \end{bmatrix},$$

$$C_8 = \begin{bmatrix} J_9 & \dots & 0 & 0 & \dots & J_{10} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_9 & 0 & \dots & J_{10} \\ 0 & \dots & 0 & J_{11} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{11} \end{bmatrix}, C_{10} = \begin{bmatrix} J_{12} & \dots & 0 & 0 & \dots & J_{10} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_{12} & 0 & \dots & J_{10} \\ 0 & \dots & 0 & J_{13} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{13} \end{bmatrix},$$

$$C_9 = \left[I_{S_2+1} \otimes \sigma_1 I_{n_1 m_1} \right], \quad J_9 = T_1 \oplus (D_0 + p_1 D_1) - (\sigma_1 + \sigma_2) I_{n_1 m_1}, \quad J_{10} = \sigma_2 I_{n_1 m_1},$$

$$J_{11} = T_1 \oplus (D_0 + p_1 D_1) - \sigma_1 I_{n_1 m_1}, \quad J_{12} = T_1 \oplus (D_0 + p_1 D_1) - \sigma_2 I_{n_1 m_1},$$

$$J_{13} = T_1 \oplus (D_0 + p_1 D_1), \quad A_{00}^{23} = \left[I_{S_1} \otimes C_{11} \quad 0 \right] \quad , C_{11} = \begin{bmatrix} 0 \\ I_{S_2} \otimes p_2 T_1^0 \otimes \beta_2 \otimes I_{m_1} \end{bmatrix}$$

,

$$A_{00}^{25} = \left[I_{S_1} \otimes C_{12} \quad 0 \right], \quad C_{12} = \begin{bmatrix} 0 \\ I_{S_2} \otimes q_2 T_1^0 \otimes I_{m_1} \end{bmatrix},$$

$$A_{00}^{33} = \begin{bmatrix} C_{13} & \dots & 0 & 0 & \dots & C_{14} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & C_{13} & 0 & \dots & C_{14} \\ 0 & \dots & 0 & C_{15} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & C_{15} \end{bmatrix}, C_{13} = \begin{bmatrix} , J_{14} & \dots & 0 & 0 & \dots & J_{15} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_{14} & 0 & \dots & J_{15} \\ 0 & \dots & 0 & J_{16} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{16} \end{bmatrix},$$

$$C_{14} = \left[I_{S_2+1} \otimes \sigma_1 I_{n_2 m_1} \right], \quad C_{15} = \begin{bmatrix} J_{17} & \dots & 0 & 0 & \dots & J_{15} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_{17} & 0 & \dots & J_{15} \\ 0 & \dots & 0 & J_{18} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{18} \end{bmatrix},$$

$$J_{14} = T_2 \oplus (D_0 + p_1 D_1) - (\sigma_1 + \sigma_2) I_{n_2 m_1} \quad , J_{16} = T_2 \oplus (D_0 + p_1 D_1) - \sigma_1 I_{n_2 m_1},$$

$$J_{15} = \sigma_2 I_{n_2 m_1}, \quad J_{17} = T_2 \oplus (D_0 + p_1 D_1) - \sigma_2 I_{n_2 m_1}, \quad J_{18} = T_2 \oplus (D_0 + p_1 D_1),$$

$$A_{00}^{35} = \left[I_{S_1+1} \otimes C_{16} \right], \quad C_{16} = \left[I_{S_2} \otimes T_2^0 \otimes I_{m_1} \quad 0 \right]$$

$$A_{00}^{44} = \begin{bmatrix} C_{17} & \dots & 0 & 0 & \dots & C_{18} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & C_{17} & 0 & \dots & C_{18} \\ 0 & \dots & 0 & C_{19} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & C_{19} \end{bmatrix}, \quad C_{17} = \begin{bmatrix} J_{19} & \dots & 0 & 0 & \dots & J_{20} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_{19} & 0 & \dots & J_{20} \\ 0 & \dots & 0 & J_{21} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{21} \end{bmatrix},$$

$$C_{18} = \left[I_{S_2+1} \otimes \sigma_1 I_{n_3 m_1} \right], \;\; C_{19} = \begin{bmatrix} J_{22} & \dots & 0 & 0 & \dots & J_{20} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_{22} & 0 & \dots & J_{20} \\ 0 & \dots & 0 & J_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{23} \end{bmatrix},$$

$$J_{19} = T_3 \oplus (D_0 + p_1 D_1) - (\sigma_1 + \sigma_2) I_{n_3 m_1}, J_{20} = \sigma_2 I_{n_3 m_1},$$

$$J_{21} = T_3 \oplus (D_0 + p_1 D_1) - \sigma_1 I_{n_3 m_1}, \;\; J_{22} = T_3 \oplus (D_0 + p_1 D_1) - \sigma_2 I_{n_3 m_1},$$

$$J_{23} = T_3 \oplus (D_0 + p_1 D_1), A_{00}^{45} = \left[I_{S_1} \otimes I_{S_2+1} \otimes T_3^0 \otimes I_{m_1} \quad 0 \right],$$

$$A_{00}^{55} = \begin{bmatrix} C_{20} & \dots & 0 & 0 & \dots & C_{21} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & C_{20} & 0 & \dots & C_{21} \\ 0 & \dots & 0 & C_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & C_{22} \end{bmatrix}, \;\; C_{20} = \begin{bmatrix} J_{24} & \dots & 0 & 0 & \dots & J_{25} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_{24} & 0 & \dots & J_{25} \\ 0 & \dots & 0 & J_{26} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{26} \end{bmatrix},$$

$$C_{21} = \left[I_{S_2+1} \otimes \sigma_1 I_{m_1} \right], \;\; C_{22} = \begin{bmatrix} J_{27} & \dots & 0 & 0 & \dots & J_{25} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_{27} & 0 & \dots & J_{25} \\ 0 & \dots & 0 & J_{28} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{28} \end{bmatrix},$$

$$J_{24} = (D_0 + p_1 D_1) - (\phi + \sigma_1 + \sigma_2) I_{m_1}, \;\; J_{26} = (D_0 + p_1 D_1) - (\phi + \sigma_1) I_{m_1},$$

$$J_{25} = \sigma_2 I_{m_1}, \;\; J_{27} = (D_0 + p_1 D_1) - (\phi + \sigma_2) I_{m_1}, \;\; J_{28} = (D_0 + p_1 D_1) - \phi I_{m_1},$$

$$A_{00}^{56} = \left[I_{S_1+1} \otimes I_{S_2+1} \otimes \phi I_{m_1} \right], \;\; A_{00}^{67} = \left[I_{S_1+1} \otimes I_{S_2+1} \otimes \eta I_{m_1} \right],$$

$$A_{00}^{71} = \begin{bmatrix} I_{S_2+1} \otimes \xi I_{m_1} & 0 \\ 0 & I_{S_1} \otimes I_{S_2+1} \otimes \alpha \otimes \xi I_{m_1} \end{bmatrix},$$

$$A_{00}^{66} = \begin{bmatrix} C_{23} & \dots & 0 & 0 & \dots & C_{21} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & C_{23} & 0 & \dots & C_{21} \\ 0 & \dots & 0 & C_{24} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & C_{24} \end{bmatrix}, \;\; A_{00}^{77} = \begin{bmatrix} C_{25} & \dots & 0 & 0 & \dots & C_{21} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & C_{25} & 0 & \dots & C_{21} \\ 0 & \dots & 0 & C_{26} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & C_{26} \end{bmatrix},$$

$$C_{23}=\begin{bmatrix} J_{29} & \dots & 0 & 0 & \dots & J_{25} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_{29} & 0 & \dots & J_{25} \\ 0 & \dots & 0 & J_{30} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{30} \end{bmatrix},\quad C_{24}=\begin{bmatrix} J_{31} & \dots & 0 & 0 & \dots & J_{25} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_{31} & 0 & \dots & J_{25} \\ 0 & \dots & 0 & J_{32} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{32} \end{bmatrix},$$

$$J_{29}=(D_0+p_1D_1)-(\eta+\sigma_1+\sigma_2)I_{m_1},\quad J_{30}=(D_0+p_1D_1)-(\eta+\sigma_1)I_{m_1},$$

$$J_{31}=(D_0+p_1D_1)-(\eta+\sigma_2)I_{m_1},\quad J_{32}=(D_0+p_1D_1)-\eta I_{m_1},$$

$$C_{25}=\begin{bmatrix} J_{33} & \dots & 0 & 0 & \dots & J_{25} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_{33} & 0 & \dots & J_{25} \\ 0 & \dots & 0 & J_{34} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{34} \end{bmatrix},\quad C_{26}=\begin{bmatrix} J_{35} & \dots & 0 & 0 & \dots & J_{25} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_{35} & 0 & \dots & J_{25} \\ 0 & \dots & 0 & J_{36} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{36} \end{bmatrix},$$

$$J_{33}=(D_0+p_1D_1)-(\xi+\sigma_1+\sigma_2)I_{m_1},\quad J_{34}=(D_0+p_1D_1)-(\xi+\sigma_1)I_{m_1},$$

$$J_{35}=(D_0+p_1D_1)-(\xi+\sigma_2)I_{m_1},\quad J_{36}=(D_0+p_1D_1)-\xi I_{m_1},$$

$$V_0=\begin{bmatrix} V_0^{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & V_0^{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & V_0^{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V_0^{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_0^{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_0^{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_0^{77} \end{bmatrix}, \tag{3}$$

$$V_0^{11}=\begin{bmatrix} I_{S_2+1}\otimes q_1D_1 & 0 \\ 0 & 0 \end{bmatrix},\quad V_0^{22}=\left[I_{S_1}\otimes I_{S_2+1}\otimes q_1D_1\otimes I_{n_1}\right],$$

$$V_0^{33}=\left[I_{S_1+1}\otimes I_{S_2}\otimes q_1D_1\otimes I_{n_2}\right],\quad V_0^{44}=\left[I_{S_1}\otimes I_{S_2+1}\otimes q_1D_1\otimes I_{n_3}\right],$$

$$V_0^{55}=\left[I_{S_1+1}\otimes I_{S_2+1}\otimes q_1D_1\right]=V_0^{66}=V_0^{77},$$

$$V_2=\begin{bmatrix} 0 & V_2^{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4}$$

$$V_2^{12}=\begin{bmatrix} 0 \\ I_{S_1}\otimes I_{S_2+1}\otimes\beta_1\otimes e_{m_2}\otimes\psi I_{m_1} \end{bmatrix},$$

$$V_1 = \begin{bmatrix} V_1^{11} & V_1^{12} & 0 & V_1^{14} & 0 & 0 & 0 \\ V_1^{21} & V_1^{22} & V_1^{23} & 0 & 0 & 0 & 0 \\ V_1^{31} & 0 & V_1^{33} & 0 & 0 & 0 & 0 \\ V_1^{41} & 0 & 0 & V_1^{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_1^{55} & V_1^{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & V_1^{66} & V_1^{67} \\ V_1^{71} & 0 & 0 & 0 & 0 & 0 & V_1^{77} \end{bmatrix}, \tag{5}$$

$$V_1^{11} = \begin{bmatrix} C_1 & 0 & \dots & 0 & 0 & \dots & C_2 \\ 0 & C_{27} & \dots & 0 & 0 & \dots & C_4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & C_{27} & 0 & \dots & C_4 \\ 0 & 0 & \dots & 0 & C_{28} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & C_{28} \end{bmatrix},$$

$$C_{27} = \begin{bmatrix} J_{37} & \dots & 0 & 0 & \dots & J_5 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_{37} & 0 & \dots & J_5 \\ 0 & \dots & 0 & J_{38} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{38} \end{bmatrix}, \quad C_{28} = \begin{bmatrix} J_{39} & \dots & 0 & 0 & \dots & J_5 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & J_{39} & 0 & \dots & J_5 \\ 0 & \dots & 0 & J_{40} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & J_{40} \end{bmatrix},$$

$$J_{37} = U \oplus D_0 - (\psi + \sigma_1 + \sigma_2) I_{m_2 m_1}, \quad J_{38} = U \oplus D_0 - (\psi + \sigma_1) I_{m_2 m_1},$$

$$J_{39} = U \oplus D_0 - (\psi + \sigma_2) I_{m_2 m_1}, \quad J_{40} = U \oplus D_0 - \psi I_{m_2 m_1},$$

$$V_1^{21} = \begin{bmatrix} C_{30} & 0 & 0 \\ 0 & I_{S_1-1} \otimes C_{31} & 0 \end{bmatrix}, \quad C_{30} = \begin{bmatrix} J_{41} & 0 \\ 0 & I_{S_2} \otimes J_{42} \end{bmatrix}, \quad C_{31} = \begin{bmatrix} , J_{43} & 0 \\ 0 & I_{S_2} \otimes J_{44} \end{bmatrix},$$

$$J_{41} = T_0^1 \otimes I_{m_1}, \quad J_{42} = q_2 T_0^1 \otimes I_{m_1}, \quad J_{43} = T_0^1 \otimes \alpha \otimes I_{m_1}, \quad J_{44} = q_2 T_0^1 \otimes \alpha \otimes I_{m_1},$$

$$V_1^{31} = \begin{bmatrix} C_{32} & 0 \\ 0 & I_{S_1} \otimes C_{33} \end{bmatrix}, \quad C_{32} = \begin{bmatrix} I_{S_2} \otimes T_2^0 \otimes I_{m_1} & 0 \end{bmatrix}, \quad C_{33} = \begin{bmatrix} I_{S_2} \otimes T_2^0 \otimes \alpha \otimes I_{m_1} & 0 \end{bmatrix},$$

$$V_1^{41} = \begin{bmatrix} I_{S_2+1} \otimes T_3^0 \otimes I_{m_1} & 0 & 0 \\ 0 & I_{S_1-1} \otimes I_{S_2+1} \otimes T_3^0 \otimes \alpha \otimes I_{m_1} & 0 \end{bmatrix},$$

$$V_1^{12} = A_{00}^{12}, \quad V_1^{14} = A_{00}^{14}, \quad V_1^{22} = A_{00}^{22}, \quad V_1^{23} = A_{00}^{23}, \quad V_1^{33} = A_{00}^{33}, \quad V_1^{44} = A_{00}^{44},$$

$$V_1^{55} = A_{00}^{55}, \quad V_1^{56} = A_{00}^{56}, \quad V_1^{66} = A_{00}^{66}, \quad V_1^{67} = A_{00}^{67}, \quad V_1^{71} = A_{00}^{71}, \quad V_1^{77} = A_{00}^{77}.$$

4. Stationary Analysis

To justify of our model, we ensure the stability of the system under certain conditions as follows:

4.1. Criteria of system stability

Let us assume that the matrix V as $V = V_0 + V_1 + V_2$ is an infinitesimal generator matrix with dimension of $\{(S_2+1)m_1 + S_1(S_2+1)m_2m_1 + S_1(S_2+1)n_1m_1 + (S_1+1)S_2n_2m_1 + S_1(S_2+1)n_3m_1 + 3(S_1+1)(S_2+1)m_1\}$,

$$V = \begin{bmatrix} V^{11} & V^{12} & 0 & V^{14} & 0 & 0 & 0 \\ V^{21} & V^{22} & V^{23} & 0 & 0 & 0 & 0 \\ V^{31} & 0 & V^{33} & 0 & 0 & 0 & 0 \\ V^{41} & 0 & 0 & V^{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V^{55} & V^{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & V^{66} & V^{67} \\ V^{71} & 0 & 0 & 0 & 0 & 0 & V^{77} \end{bmatrix}, \tag{6}$$

where

$$V^{11} = V_1^{11} + V_0^{11},\ V^{12} = V_1^{12} + V_2^{12},\ V^{14} = V_1^{14},\ V^{21} = V_1^{21},\ V^{22} = V_1^{22} + V_0^{22},\ V^{23} = V_1^{23},$$

$$V^{31} = V_1^{31},\ V^{33} = V_1^{33} + V_0^{33},\ V^{41} = V_1^{41},\ V^{44} = V_1^{44} + V_0^{44},\ V^{55} = V_1^{55} + V_0^{55},$$

$$V^{56} = V_1^{56},\ V^{66} = V_1^{66} + V_0^{66},\ V^{67} = V_1^{67},\ V^{71} = V_1^{71},\ V^{77} = V_1^{77} + V_0^{77}.$$

Let $\boldsymbol{\zeta}$ be the invariant probability vector can be partitioned as $(\boldsymbol{\zeta_0}, \boldsymbol{\zeta_1}, \boldsymbol{\zeta_2}, \boldsymbol{\zeta_3}, \boldsymbol{\zeta_4}, \boldsymbol{\zeta_5}, \boldsymbol{\zeta_6})$ with the dimensions $\{(S_2+1)m_1+S_1(S_2+1)m_2m_1, S_1(S_2+1)n_1m_1, (S_1+1)S_2n_2m_1, S_1(S_2+1)n_3m_1, (S_1+1)(S_2+1)m_1, (S_1+1)(S_2+1)m_1, (S_1+1)(S_2+1)m_1\}$ of the matrix V will be obtained the values such that it must be satisfied these conditions $\boldsymbol{\zeta}V = 0$ and $\boldsymbol{\zeta}e = 1$,

$$\begin{aligned} \text{i.e., } \boldsymbol{\zeta_0}V^{11} + \boldsymbol{\zeta_1}V^{21} + \boldsymbol{\zeta_2}V^{31} + \boldsymbol{\zeta_3}V^{41} + \boldsymbol{\zeta_6}V^{71} &= 0, & (7)\\ \boldsymbol{\zeta_0}V^{12} + \boldsymbol{\zeta_1}V^{22} &= 0, & (8)\\ \boldsymbol{\zeta_1}V^{23} + \boldsymbol{\zeta_2}V^{33} &= 0, & (9)\\ \boldsymbol{\zeta_0}V^{14} + \boldsymbol{\zeta_3}V^{44} &= 0, & (10)\\ \boldsymbol{\zeta_4}V^{55} &= 0, & (11)\\ \boldsymbol{\zeta_4}V^{56} + \boldsymbol{\zeta_5}V^{66} &= 0, & (12)\\ \boldsymbol{\zeta_5}V^{67} + \boldsymbol{\zeta_6}V^{77} &= 0, & (13) \end{aligned}$$

along with normalising condition that,

$$\Bigg[\sum_{j_2=0}^{S_2} \zeta_{0,0,j_2} e_{m_1} + \sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2} \zeta_{0,j_1,j_2} e_{m_1m_2} + \sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2} \zeta_{1,j_1,j_2} e_{m_1n_1}$$
$$+ \sum_{j_1=0}^{S_1}\sum_{j_2=1}^{S_2} \zeta_{2,j_1,j_2} e_{m_1n_2} + \sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2} \zeta_{3,j_1,j_2} e_{m_1n_3}$$

$$+\sum_{j_1=0}^{S_1}\sum_{j_2=0}^{S_2}\zeta_{4,j_1,j_2}e_{m_1}+\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\zeta_{5,j_1,j_2}e_{m_1}+\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\zeta_{6,j_1,j_2}e_{m_1}\Bigg] \;=\; 1. \tag{14}$$

From the equations (11) to (13) we obtained $\boldsymbol{\zeta_4}=0,\ \boldsymbol{\zeta_5}=0$ and $\boldsymbol{\zeta_6}=0.$ Thus the equation (14) is rewritten as

$$\Bigg[\sum_{j_2=0}^{S_2}\zeta_{0,0,j_2}e_{m_1}+\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\zeta_{0,j_1,j_2}e_{m_1m_2}+\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\zeta_{1,j_1,j_2}e_{m_1n_1}$$
$$+\sum_{j_1=0}^{S_1}\sum_{j_2=1}^{S_2}\zeta_{2,j_1,j_2}e_{m_1n_2}+\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\zeta_{3,j_1,j_2}e_{m_1n_3}\Bigg] \;=\; 1. \tag{15}$$

Since the LIQBD process of with infinitesimal generator Q is stable if and only if the left drift rate exceeds the right drift rate. That is, $\zeta V_0 e<\zeta V_2 e$, acccording to [19].

i.e.,$(\boldsymbol{\zeta_0}V_0^{11})e_{m_1+m_1m_2}+(\boldsymbol{\zeta_1}V_0^{22})e_{m_1n_1}+(\boldsymbol{\zeta_2}V_0^{33})e_{m_1n_2}+(\boldsymbol{\zeta_3}V_0^{44})e_{m_1n_3}$
$$+(\boldsymbol{\zeta_4}V_0^{55})e_{m_1}+(\boldsymbol{\zeta_5}V_0^{66})e_{m_1}+(\boldsymbol{\zeta_6}V_0^{77})e_{m_1} \;<\; (\boldsymbol{\zeta_1}V_2^{12})e_{m_1n_1}$$

$$(\boldsymbol{\zeta_0}V_0^{11})e_{m_1+m_1m_2}+(\boldsymbol{\zeta_1}V_0^{22})e_{m_1n_1}+(\boldsymbol{\zeta_2}V_0^{33})e_{m_1n_2}+(\boldsymbol{\zeta_3}V_0^{44})e_{m_1n_3} \;<\; (\boldsymbol{\zeta_1}V_2^{12})e_{m_1n_1} \tag{16}$$

After replacing the entries of sub blocks of V_0 and V_2 also simplifying then finally we get

$$\Bigg[\sum_{j_2=0}^{S_2}\zeta_{0,0,j_2}q_1D_1e_{m_1}+\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\zeta_{1,j_1,j_2}\left[e_{n_1}\otimes q_1D_1e_{m_1}\right]+\sum_{j_1=0}^{S_1}\sum_{j_2=1}^{S_2}\zeta_{2,j_1,j_2}\left[e_{n_2}\otimes q_1D_1e_{m_1}\right]$$
$$+\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\zeta_{3,j_1,j_2}\left[e_{n_3}\otimes q_1D_1e_{m_1}\right]\Bigg]<\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\zeta_{0,j_1,j_2}\left[e_{n_1}\beta_1\otimes e_{m_2}\otimes\psi e_{m_1}\right] \tag{17}$$

4.2. Analysis of stationary probability vector

Assuming that x represents the invariant probability vector, it is possible to divide it as $(x_0,x_1,x_2,\dots)$ with each of its dimensions are $\{(S_2+1)m_1+S_1(S_2+1)m_2m_1+S_1(S_2+1)n_1m_1+(S_1+1)S_2n_2m_1+S_1(S_2+1)n_3m_1+3(S_1+1)(S_2+1)m_1\}$ of the matrix Q will be obtained the values such that it must be satisfied these conditions $xQ=0$ and $xe=1.$

After the stability criterion has been met, we are able to determine the probability vector for the steady state by applying the equations that are listed below.

$$x_i \;=\; x_0\,R^i,\ i\geq 1. \tag{18}$$

The non-negative smallest solution R, which is based on [20] work, is able to satisfy the matrix quadratic equation $R^2V_2+RV_1+V_0=0.$ The rate matrix is obtained by solving the matrix quadratic equation. The order of the rate matrix R is determined by the equation

$\{(S_2+1)m_1+S_1(S_2+1)m_2m_1+S_1(S_2+1)n_1m_1+(S_1+1)S_2n_2m_1+S_1(S_2+1)n_3m_1+3(S_1+1)(S_2+1)m_1\}$, and it satisfies the constraint $RV_2e = V_0e$.

The sub vector x_0 can be determined by using the following equations:

$$x_0(A_{00} + RV_2) \quad = \quad 0, \tag{19}$$

with the normalizing property that,

$$x_0(I - R)^{-1}e_0 = 1, \tag{20}$$

where e_0 is the column vector of order $(S_2+1)m_1+S_1(S_2+1)m_2m_1+S_1(S_2+1)n_1m_1+(S_1+1)S_2n_2m_1+S_1(S_2+1)n_3m_1+3(S_1+1)(S_2+1)m_1$. The R matrix can be generated analytically by using key steps in the logarithmic reduction process, described by [16]

Step 1: $H \leftarrow (-V_1)^{-1}V_0$, $L \leftarrow (-V_1)^{-1}V_2$, $G = L$ and $T = H$.

Step 2: $U = HL + LH$; $M = H^2$; $H \leftarrow (I - U)^{-1}M$; $M \leftarrow L^2$;
$L \leftarrow (I - U)^{-1}M$; $G \leftarrow G + TL$; $T \leftarrow TH$
continue Step 1 until $||e - Ge||_\infty < \epsilon$

Step 3: $R = -V_0(V_1 + V_0G)^{-1}$.

5. Busy Period Analysis

- To evaluate the active period, we consider the time interval beginning with the arrival of a customer to an empty system and ending when the system returns to the empty state. During this period, the process moves from level 0 to level 1 and subsequently returns from level 1 to level 0. Thus, the busy cycle comprises the initial movement to level 1 and the subsequent sojourn through one or more higher levels before the system returns to level 0.
- Before determining the active period, we first provide an overview of the fundamental epoch. In the QBD process, the initial transition occurs when the process moves from level i to level $i-1$, for $i \geq 1$.
- It is necessary to examine the boundary state possibilities that correspond to $i = 0, 1$ independently. For every level i with $i \geq 1$, there are $\{(S_2+1)m_1+S_1(S_2+1)m_2m_1+S_1(S_2+1)n_1m_1+(S_1+1)S_2n_2m_1+S_1(S_2+1)n_3m_1+3(S_1+1)(S_2+1)m_1\}$ states that correspond and need to be taken into account. This means that the state (i, j) denotes the jth state associated with level i. The states are arranged according to lexicographic ordering.
- The quantity $\tilde{K}(c, s)$ represents the transition of the process from state (i, j) to state (i, j') through a leftward movement. Consider a QBD process that starts in state (i, j) at time $t = 0$ and reaches level $i - 1$ for the first time at time w. The corresponding joint transform is first defined, following [19], as:

$$\tilde{K}_{jj'}(c, s) = \sum_{i=1}^{\infty} c^i \int_0^{\infty} e^{-sx} dK_{jj'}(u, x); |c| \leq 1, Re(s) \geq 0, \tag{21}$$

When the matrix $\tilde{K}(c,s)$ is expressed in terms of its elements as $\tilde{K}(c,s) = \tilde{K}_{jj'}(c,s)$, each element represents the corresponding transition from state (i,j) to state (i,j'). The previously defined matrix $\tilde{K}(c,s)$ therefore satisfies the following equation:

$$\tilde{K}(c,s) = w(sI - V_1)^{-1}V_2 + (sI - V_1)^{-1}V_0\tilde{K}^2(c,s). \tag{22}$$

- Disregarding the boundary states, the initial passage time can be determined using the matrix $K = K_{jj'} = \tilde{K}(1,0)$. Given the rate matrix $R = -(V_0 + R^2V_2)V_1^{-1}$, the matrix K can subsequently be obtained as follows:

$$K = -(V_1 + RV_2)^{-1}V_2. \tag{23}$$

  Alternatively, the values of the K matrix could be found using the concept of logarithmic reduction technique.
- The following equations relate to boundary levels 1 and 0 : $\tilde{K}^{(1,0)}(c,s)$, $\tilde{K}^{(0,0)}(c,s)$

$$\tilde{K}^{(1,0)}(c,s) = w(sI - V_1)^{-1}V_2 + (sI - V_1)^{-1}V_0\tilde{K}(c,s)\tilde{K}^{(1,0)}(c,s), \tag{24}$$
$$\tilde{K}^{(0,0)}(c,s) = (sI - A_{00})^{-1}V_0\tilde{K}(c,s). \tag{25}$$

According to the stochastic nature of K, $\tilde{K}^{(0,0)}(1,0)$ and $\tilde{K}^{(1,0)}(1,0)$, the matrices are employed in the computation of the ensuing scenarios. The moments that can be determined are as follows:

$$\vec{\mathfrak{K}}_1 = -\frac{\partial}{\partial s}\tilde{K}(c,s)\Big|_{c=1,s=0} e = -[V_1 + V_0(I+K)]^{-1}e, \tag{26}$$

$$\vec{\mathfrak{K}}_2 = \frac{\partial}{\partial c}\tilde{K}(c,s)\Big|_{c=1,s=0} e = -[V_1 + V_0(I+K)]^{-1}V_2e, \tag{27}$$

$$\vec{\mathfrak{K}}_1^{(1,0)} = -\frac{\partial}{\partial s}\tilde{K}^{(1,0)}(c,s)\Big|_{c=1,s=0} e = -[V_1 + V_0K]^{-1}(V_0\vec{\mathfrak{K}}_1 + e), \tag{28}$$

$$\vec{\mathfrak{K}}_2^{(1,0)} = \frac{\partial}{\partial c}\tilde{K}^{(1,0)}(c,s)\Big|_{c=1,s=0} e = -[V_1 + V_0K]^{-1}(V_0\vec{\mathfrak{K}}_2 + V_2e), \tag{29}$$

$$\vec{\mathfrak{K}}_1^{(0,0)} = \frac{\partial}{\partial s}\tilde{K}^{(0,0)}(c,s)\Big|_{c=1,s=0} e = -A_{00}^{-1}[V_0\vec{\mathfrak{K}}_1^{(1,0)} + e], \tag{30}$$

$$\vec{\mathfrak{K}}_2^{(0,0)} = \frac{\partial}{\partial c}\tilde{K}^{(0,0)}(c,s)\Big|_{c=1,s=0} e = -A_{00}^{-1}[V_0\vec{\mathfrak{K}}_2^{(1,0)}]. \tag{31}$$

6. System Performance Measures

We calculate some system performance measures for our constructive model. The notation $x_{iur_1r_2k_1k_2}$ indicates that the steady state probability of the number of customers in orbit is i, server status is u, number of service phase is r_1 for incoming client, number of service phase is r_2 for outgoing client, number of incoming client phase is k_1 and number of outgoing client phase is k_2.

- Probability of a server is idle:

$$P_{IDLE} = \sum_{i=0}^{\infty}\sum_{j_2=0}^{S_2}\sum_{k_1=1}^{m_1} x_{i0j_2k_1}$$

- Probability of a server makes outgoing call arrivals:

$$P_{SMOCA} = \sum_{i=0}^{\infty}\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\sum_{k_2=1}^{m_2}\sum_{k_1=1}^{m_1} x_{i0j_1j_2k_2k_1}$$

- Probability that a server busy over incoming call arrivals:

$$P_{SBICA} = \sum_{i=0}^{\infty}\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\sum_{r_1=1}^{n_1}\sum_{k_1=1}^{m_1} x_{i1j_1j_2r_1k_1}$$

- Probability that a server busy over optional service:

$$P_{SBOPS} = \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=1}^{S_2}\sum_{r_2=1}^{n_2}\sum_{k_1=1}^{m_1} x_{i2j_1j_2r_2k_1}$$

- Probability of a server is busy with outgoing call arrivals:

$$P_{SBOCA} = \sum_{i=0}^{\infty}\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\sum_{r_3=1}^{n_3}\sum_{k_1=1}^{m_1} x_{i3j_1j_2r_3k_1}$$

- Probability of a server is close down:

$$P_{SCD} = \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=0}^{S_2}\sum_{k_1=1}^{m_1} x_{i4j_1j_2k_1}$$

- Probability of a server under vacation:

$$P_{SV} = \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=0}^{S_2}\sum_{k_1=1}^{m_1} x_{i5j_1j_2k_1}$$

- Probability of a server is setup:

$$P_{SS} = \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=0}^{S_2}\sum_{k_1=1}^{m_1} x_{i6j_1j_2k_1}$$

- Average count of customers in the orbit:

$$E_{orbit} = \sum_{i=0}^{\infty} i x_i e$$

- Probability that server under busy:

$$P_{BS} \quad = \quad P_{SBICA} + P_{SBOPS} + P_{SBOCA}$$

- Average size of system:

$$E_{system} \quad = \quad E_{orbit} + P_{BS}$$

- Mean orbit size during the server makes outgoing call arrivals:

$$E_{SMOCA} \quad = \quad \sum_{i=0}^{\infty} \sum_{j_1=1}^{S_1} \sum_{j_2=0}^{S_2} \sum_{k_2=1}^{m_2} \sum_{k_1=1}^{m_1} i x_{i0j_1j_2k_2k_1}$$

- Mean orbit size during the server is busy over incoming call arrivals:

$$E_{SBICA} \quad = \quad \sum_{i=0}^{\infty} \sum_{j_1=1}^{S_1} \sum_{j_2=0}^{S_2} \sum_{r_1=1}^{n_1} \sum_{k_1=1}^{m_1} i x_{i1j_1j_2r_1k_1}$$

- Mean orbit size during the server is busy over optional service:

$$E_{SBOPS} \quad = \quad \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=1}^{S_2} \sum_{r_2=1}^{n_2} \sum_{k_1=1}^{m_1} i x_{i2j_1j_2r_2k_1}$$

- Mean orbit size during the server busy with outgoing call arrivals:

$$E_{SBOCA} \quad = \quad \sum_{i=0}^{\infty} \sum_{j_1=1}^{S_1} \sum_{j_2=0}^{S_2} \sum_{r_3=1}^{n_3} \sum_{k_1=1}^{m_1} i x_{i3j_1j_2r_3k_1}$$

- Mean orbit size during the server is close down:

$$E_{SCD} \quad = \quad \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{S_2} \sum_{k_1=1}^{m_1} i x_{i4j_1j_2k_1}$$

- Mean orbit size during the server under vacation:

$$E_{SV} \quad = \quad \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{S_2} \sum_{k_1=1}^{m_1} i x_{i5j_1j_2k_1}$$

- Mean orbit size during the server is setup:

$$E_{SS} \quad = \quad \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{S_2} \sum_{k_1=1}^{m_1} i x_{i6j_1j_2k_1}$$

- Expected rate of replenishment of essential items:

$$E_{RE} = \sigma_1 \left[\sum_{i=0}^{\infty} \sum_{j_1=0}^{s_1} \sum_{j_2=0}^{S_2} x_{i0j_1j_2}e + \sum_{i=0}^{\infty} \sum_{j_1=1}^{s_1} \sum_{j_2=0}^{S_2} x_{i1j_1j_2}e + \sum_{i=0}^{\infty} \sum_{j_1=0}^{s_1} \sum_{j_2=1}^{S_2} x_{i2j_1j_2}e \right.$$
$$+ \sum_{i=0}^{\infty} \sum_{j_1=1}^{s_1} \sum_{j_2=0}^{S_2} x_{i3j_1j_2}e + \sum_{i=0}^{\infty} \sum_{j_1=0}^{s_1} \sum_{j_2=0}^{S_2} x_{i4j_1j_2}e + \sum_{i=0}^{\infty} \sum_{j_1=0}^{s_1} \sum_{j_2=0}^{S_2} x_{i5j_1j_2}e$$
$$\left. + \sum_{i=0}^{\infty} \sum_{j_1=0}^{s_1} \sum_{j_2=0}^{S_2} x_{i6j_1j_2}e \right]$$

- Expected rate of replenishment of optional items:

$$E_{RO} = \sigma_2 \left[\sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{s_2} x_{i0j_1j_2}e + \sum_{i=0}^{\infty} \sum_{j_1=1}^{S_1} \sum_{j_2=0}^{s_2} x_{i1j_1j_2}e + \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=1}^{s_2} x_{i2j_1j_2}e \right.$$
$$+ \sum_{i=0}^{\infty} \sum_{j_1=1}^{S_1} \sum_{j_2=0}^{s_2} x_{i3j_1j_2}e + \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{s_2} x_{i4j_1j_2}e + \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{s_2} x_{i5j_1j_2}e$$
$$\left. + \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{s_2} x_{i6j_1j_2}e \right]$$

- Expected loss rate of customers:

$$E_B = p_1\lambda_1 \left[\sum_{i=0}^{\infty} \sum_{j_2=0}^{S_2} \sum_{k_1=1}^{m_1} x_{i00j_2k_1} + \sum_{i=0}^{\infty} \sum_{j_1=1}^{S_1} \sum_{j_2=0}^{S_2} \sum_{r_1=1}^{n_1} \sum_{k_1=1}^{m_1} x_{i1j_1j_2r_1k_1} \right.$$
$$+ \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=1}^{S_2} \sum_{r_2=1}^{n_2} \sum_{k_1=1}^{m_1} x_{i2j_1j_2r_2k_1} + \sum_{i=0}^{\infty} \sum_{j_1=1}^{S_1} \sum_{j_2=0}^{S_2} \sum_{r_3=1}^{n_3} \sum_{k_1=1}^{m_1} x_{i3j_1j_2r_3k_1}$$
$$+ \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{S_2} \sum_{k_1=1}^{m_1} x_{i4j_1j_2k_1} + \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{S_2} \sum_{k_1=1}^{m_1} x_{i5j_1j_2k_1}$$
$$\left. + \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{S_2} \sum_{k_1=1}^{m_1} x_{i6j_1j_2k_1} \right]$$

- Expected server is in setup:

$$E_S = \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{S_2} \sum_{k_1=1}^{m_1} x_{i6j_1j_2k_1}$$

- Expected server is in vacation:

$$E_V = \sum_{i=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{S_2} \sum_{k_1=1}^{m_1} x_{i5j_1j_2k_1}$$

- Mean Essential Inventory level:

$$
\begin{aligned}
E_{EIL} = & \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=0}^{S_2}\sum_{k_2=1}^{m_2}\sum_{k_1=1}^{m_1} j_1 x_{i0j_1j_2k_2k_1} + \sum_{i=0}^{\infty}\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\sum_{r_1=1}^{n_1}\sum_{k_1=1}^{m_1} j_1 x_{i1j_1j_2r_1k_1} \\
& + \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=1}^{S_2}\sum_{r_2=1}^{n_2}\sum_{k_1=1}^{m_1} j_1 x_{i2j_1j_2r_2k_1} + \sum_{i=0}^{\infty}\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\sum_{r_3=1}^{n_3}\sum_{k_1=1}^{m_1} j_1 x_{i3j_1j_2r_3k_1} \\
& + \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=0}^{S_2}\sum_{k_1=1}^{m_1} j_1 x_{i4j_1j_2k_1} + \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=0}^{S_2}\sum_{k_1=1}^{m_1} j_1 x_{i5j_1j_2k_1} \\
& + \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=0}^{S_2}\sum_{k_1=1}^{m_1} j_1 x_{i6j_1j_2k_1}
\end{aligned}
$$

- Mean Optional Inventory level:

$$
\begin{aligned}
E_{OIL} = & \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=0}^{S_2}\sum_{k_2=1}^{m_2}\sum_{k_1=1}^{m_1} j_2 x_{i0j_1j_2k_2k_1} + \sum_{i=0}^{\infty}\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\sum_{r_1=1}^{n_1}\sum_{k_1=1}^{m_1} j_2 x_{i1j_1j_2r_1k_1} \\
& + \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=1}^{S_2}\sum_{r_2=1}^{n_2}\sum_{k_1=1}^{m_1} j_2 x_{i2j_1j_2r_2k_1} + \sum_{i=0}^{\infty}\sum_{j_1=1}^{S_1}\sum_{j_2=0}^{S_2}\sum_{r_3=1}^{n_3}\sum_{k_1=1}^{m_1} j_2 x_{i3j_1j_2r_3k_1} \\
& + \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=0}^{S_2}\sum_{k_1=1}^{m_1} j_2 x_{i4j_1j_2k_1} + \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=0}^{S_2}\sum_{k_1=1}^{m_1} j_2 x_{i5j_1j_2k_1} \\
& + \sum_{i=0}^{\infty}\sum_{j_1=0}^{S_1}\sum_{j_2=0}^{S_2}\sum_{k_1=1}^{m_1} j_2 x_{i6j_1j_2k_1}
\end{aligned}
$$

7. Expected Total Cost

We discuss the cost of some performance measures of the system under consideration. Hence we introduce the expected total cost of system per unit time (ETC) is given below.

$$
\begin{aligned}
ETC \quad = \quad & [C_S E_{orbit} + C_{EH}E_{EIL} + C_{OH}E_{OIL} + [K_1 + C_{R_1}(S_1 - s_1)]E_{RE} \\
& +[K_2 + C_{R_2}(S_2 - s_2)]E_{RO} + C_B E_B + C_S E_S + C_V E_V] \,, \qquad (32)
\end{aligned}
$$

where

- C_S describes holding cost of a customer in the system per unit time
- C_{EH} describes holding cost of inventory(essential) I per unit time
- C_{OH} describes holding cost of inventory(optional) II per unit time
- K_1 describes fixed cost of delivery service of essential inventories
- C_{R_1} describes the carriage cost of the delivery service per essential inventory
- K_2 describes fixed cost of delivery service of optional inventories

- C_{R_2} describes the carriage cost of the delivery service per optional inventory
- C_B describes cost incurred for each lost customer per unit time
- C_S describes setup cost per unit time
- C_V describes vacation cost per unit time

8. Numerical Results

In the section that follows, we will be analysing the behaviour of the models by making use of both numerical and graphical demonstrations. The following three representations illustrate different aspects of the MAP, and each maintains a consistent mean value of 1 across all arrival processes. According to the research conducted by [9] , the three arrival value sets were used as input data in their literature.

- **Arrival Erlang (A-EL):**

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} \quad D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix},$$

- **Arrival Exponential (A-EP):**

$$D_0 = \begin{bmatrix} -1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 \end{bmatrix},$$

- **Arrival Hyper-exponential (A-HEX):**

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix},$$

The outgoing arrivals and services have three distinct phase-type distributions and the following notations A-EP, A-EL, A-HEX, S-EL, S-EP and S-HEX respectively for exponential, Erlang and hyper-exponential cases under consideration.

- **Service Erlang (S-EL):**

$$\alpha = \beta_1 = \beta_2 = (1, 0), \; R = T_1 = T_2 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}.$$

- **Service Exponential (S-EP):**

$$\alpha = \beta_1 = \beta_2 = [1], \; R = T_1 = T_2 = [-1].$$

- **Service Hyper-exponential (S-HEX):**

$$\alpha = \beta_1 = \beta_2 = (0.8, 0.2), \; R = T_1 = T_2 = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}.$$

Example 8.1. We explored the effects of vacation rate (η), probability of server vacation and number of consumers in the orbit (E_{orbit}). In order to attain system stability, we fix $\lambda = 1.5$, $\mu_1 = 6$, $\mu_2 = 5$, $\mu_3 = 4$, $\mu_4 = 1$, $\sigma_1 = 3$, $\sigma_2 = 2$, $s_1 = 6$, $s_2 = 4$, $S_1 = 12$, $S_2 = 8$, $\psi = 60$, $\phi = 2$, $\xi = 2$, $p_1 = 0.4$, $q_1 = 0.6$, $p_2 = 0.7$, $q_2 = 0.3$.

- We combine the arrival and service time categories in Tables 3 through 5 to investigate the vacation rate (η), probability of server vacation versus number of consumers in the orbit (E_{orbit}).
- An increase in the vacation rate (η) leads to a reduction in the probability that the server is in the vacation state, as well as a decrease in the average number of customers in the orbit.
- When comparing service timings to those of all other arrivals, the number of consumers in the orbit, denoted by the symbol E_{orbit}, decreases rapidly for Erlang service, but it decreases slowly for hyper-exponential service.

Example 8.2. We explored the effects of retrial rate(ψ) versus the number of consumers in the orbit (E_{orbit}). In order to attain system stability, we fix $\lambda = 1.5$, $\mu_1 = 6$, $\mu_2 = 5$, $\mu_3 = 4$, $\mu_4 = 1$, $\sigma_1 = 3$, $\sigma_2 = 2$, $s_1 = 6$, $s_2 = 4$, $S_1 = 12$, $S_2 = 8$, $\phi = 2$, $\xi = 2$, $p_1 = 0.4$, $q_1 = 0.6$, $p_2 = 0.7$, $q_2 = 0.3$.

- We combine the arrival and service time categories in Figures 2 through 4 to investigate the retrial rate versus the number of consumers in the orbit.
- When the retrial rate (ψ) rises, the corresponding the number of consumers in the orbit (E_{orbit}) reduces.
- When the service times of hyper-exponential service are compared to those of all other services, the (E_{orbit}) decreases almost immediately, but the decrease for Erlang service is more gradual. In a similar vein, when it comes to service durations, the (E_{orbit}) diminishes at a slower rate in Erlang services compared to hyper-exponential services.

Example 8.3. We explored the effects of vacation rate (η) and service rate of the incoming calls (μ_1) versus the average number of consumers in the system (E_{system}). In order to attain system stability, we fix $\lambda = 1.5$, $\mu_2 = 5$, $\mu_3 = 4$, $\mu_4 = 1$, $\sigma_1 = 3$, $\sigma_2 = 2$, $s_1 = 6$, $s_2 = 4$, $S_1 = 12$, $S_2 = 8$, $\psi = 60$, $\phi = 2$, $\xi = 2$, $p_1 = 0.4$, $q_1 = 0.6$, $p_2 = 0.7$, $q_2 = 0.3$.

- We combine the arrival and service time categories to investigate the vacation rate (η) and service rate of the incoming calls (μ_1) versus the average number of consumers in the system, using Figures 5 through 13.
- When the vacation rate (η) and service rate of the incoming calls (μ_1) increases the corresponding average number of consumers in the system (E_{system}) decreases.
- When the arrival times of hyper-exponential arrivals are compared to those of all other arrivals, the E_{orbit} decreases rapidly, whereas the E_{orbit} decreases quite slowly for Erlang arrivals.

9. Optimum Cost

Tables 1 and 2 present the effect of varying the values of (s_1, S_1) (s_2, S_2) on the expected total cost of the system. These cost measures help in formulating appropriate control policies aimed at reducing the overall system cost. It has been computed by considering for the Poisson arrivals and service time follows exponential distribution.

Table 1. Performance of the Cost Function for Various (s_1, S_1) Levels

$S_1 \downarrow s_1 \rightarrow$	2	3	4	5	6
9	38.004	40.850	44.044	47.919	53.130
10	37.758	40.348	43.147	46.321	50.190
11	**37.672**	40.083	42.632	45.422	48.591
12	37.694	**39.981**	42.366	44.910	47.691
13	37.787	39.988	**42.259**	44.637	47.178
14	37.924	40.068	42.262	**44.529**	46.906
15	38.089	40.194	42.338	44.531	**46.797**
16	38.268	40.349	42.461	44.606	46.799
17	38.454	40.519	42.613	44.729	46.873

Table 2. Performance of the Cost Function for Various (s_2, S_2) Levels

$S_2 \downarrow s_2 \rightarrow$	2	3	4	5	6
7	38.731	40.904	43.678	47.30	53.130
8	38.507	40.384	42.574	45.353	49.507
9	**38.430**	40.161	42.054	44.249	47.030
10	38.449	**40.090**	41.832	43.730	45.926
11	38.521	40.112	**41.763**	43.508	45.407
12	38.624	40.187	41.786	**43.440**	45.185
13	38.742	40.291	41.862	43.462	**45.117**
14	38.862	40.409	41.966	43.538	45.140
15	38.976	40.530	42.084	43.643	45.216

Table 3. Vacation rate(η) Vs P_{SV} and E_{orbit} $-$ A-EP

	S-EP		S-EL		S-HEX	
η	E_{orbit}	P_{SV}	E_{orbit}	P_{SV}	E_{orbit}	P_{SV}
4.1	1.7815247	0.1222761	1.7625395	0.1222790	1.9069956	0.1222546
4.2	1.7781871	0.1197319	1.7591814	0.1197348	1.9037935	0.1197108
4.3	1.7750257	0.1172908	1.7560002	0.1172936	1.9007625	0.1172702
4.4	1.7720271	0.1149468	1.7529826	0.1149495	1.8978896	0.1149266
4.5	1.7691794	0.1126941	1.7501166	0.1126968	1.8951631	0.1126743
4.6	1.7664716	0.1105276	1.7473912	0.1105302	1.8925724	0.1105082
4.7	1.7638940	0.1084424	1.7447964	0.1084450	1.8901077	0.1084234
4.8	1.7614375	0.1064341	1.7423234	0.10643662	1.8877603	0.1064154
4.9	1.7590940	0.1044985	1.7399639	0.1045009	1.8855223	0.1044801

In order to obtain optimum cost of this model, we fix the values, $C_S = 2$, $C_{EH} = 4$, $C_{OH} = 3$, $K_1 = 80$, $C_{R_1} = 10$, $K_2 = 60$, $C_{R_2} = 5$, $C_B = 20$, $C_S = 15$, $C_V = 12$, $\lambda = 1.5$, $\mu_1 = 6$, $\mu_2 = 5$, $\mu_3 = 4$, $\mu_4 = 1$, $\sigma_1 = 3$, $\sigma_2 = 2$, $\psi = 60$, $\phi = 2$, $\xi = 2$, $p_1 = 0.4$, $q_1 = 0.6$, $p_2 = 0.7$, $q_2 = 0.3$, also in addition with, we considered $s_2 = 1$, $S_2 = 5$ for Table 1 and $s_1 = 2$, $S_1 = 11$ for Table 2.

Hence from the tables 1 and 2 we infer that, the optimum expected cost is obtained for the various levels of the essential and optional inventories.

Table 4. Vacation rate(η) Vs P_{SV} and E_{orbit} − A-EL

	S-EP		S-EL		S-HEX	
η	E_{orbit}	P_{SV}	E_{orbit}	P_{SV}	E_{orbit}	P_{SV}
4.1	1.7282265	0.1265542	1.7144628	0.1266419	1.8360047	0.1263083
4.2	1.7247489	0.1239388	1.7109796	0.1240256	1.8326117	0.1236951
4.3	1.7214525	0.1214285	1.7076777	0.1215145	1.8293971	0.1211872
4.4	1.7183238	0.1190173	1.7045436	0.1191025	1.8263475	0.1187783
4.5	1.7153506	0.1166994	1.7015652	0.1167838	1.8234507	0.1164626
4.6	1.7125217	0.1144695	1.6987312	0.1145532	1.8206959	0.1142350
4.7	1.7098270	0.1123227	1.6960315	0.1124056	1.8180729	0.1120905
4.8	1.7072574	0.1102545	1.6934567	0.1103367	1.8155727	0.1100245
4.9	1.7048045	0.1082607	1.6909993	0.1083422	1.8131871	0.1080329

Table 5. Vacation rate(η) Vs P_{SV} and E_{orbit} − A-HEX

	S-EP		S-EL		S-HEX	
η	E_{orbit}	P_{SV}	E_{orbit}	P_{SV}	E_{orbit}	P_{SV}
4.1	2.0084297	0.1072586	1.9716126	0.1070780	2.2210119	0.1080791
4.2	2.0055494	0.1050076	1.9686865	0.1048298	2.2183980	0.1058158
4.3	2.0028289	0.1028491	1.9659217	0.1026739	2.2159347	0.1036452
4.4	2.0002558	0.1007774	1.9633058	0.1006048	2.2136100	0.1015619
4.5	1.9978188	0.0987875	1.9608275	0.0986174	2.2114133	0.0995606
4.6	1.9955079	0.0968746	1.9584766	0.0967069	2.2093348	0.0976366
4.7	1.9933140	0.0950342	1.9562439	0.0948690	2.2073657	0.0957854
4.8	1.9912287	0.0932625	1.9541210	0.0930996	2.2054981	0.0940031
4.9	1.9892444	0.0915555	1.9521004	0.0913949	2.2037247	0.0922859

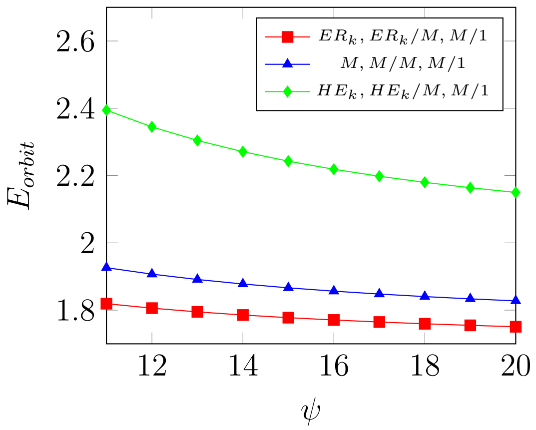

Figure 2. Retrial rate(ψ) Vs E_{orbit} - EXP−S

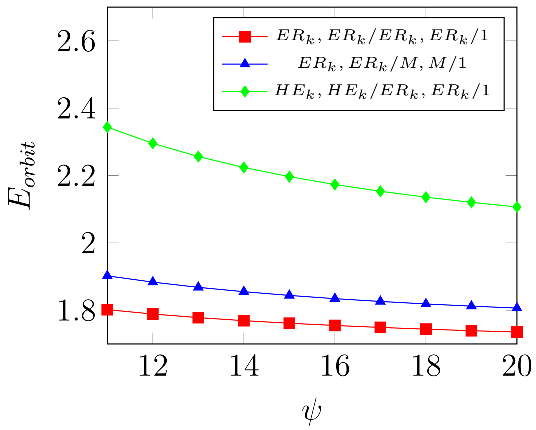

Figure 3. Retrial rate(ψ) Vs E_{orbit}-ER−S

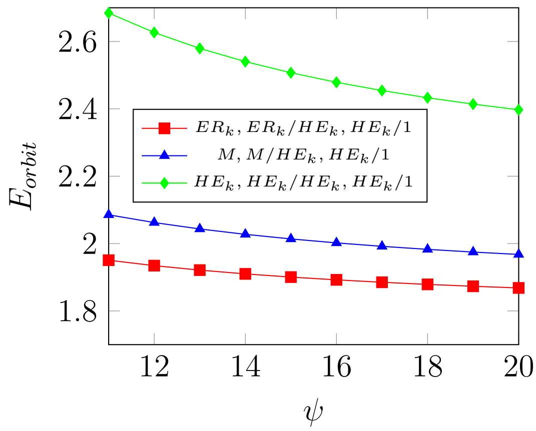

Figure 4. Retrial rate(ψ) Vs E_{orbit} - HEX−S

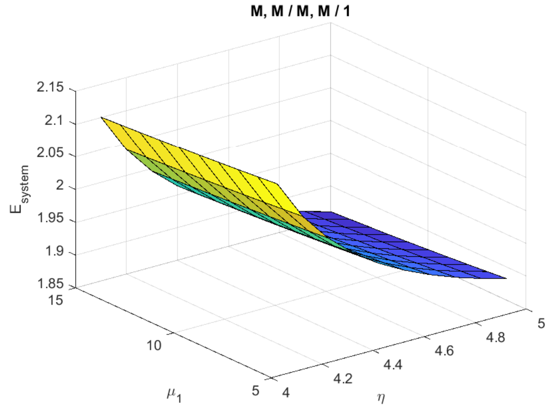

Figure 5. $M, M/M, M/1$ - η, μ_1 vs E_{system}

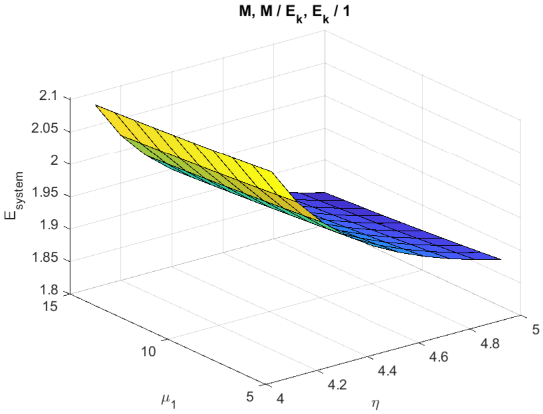

Figure 6. $M, M/E_k, E_k/1$ - η, μ_1 vs E_{system}

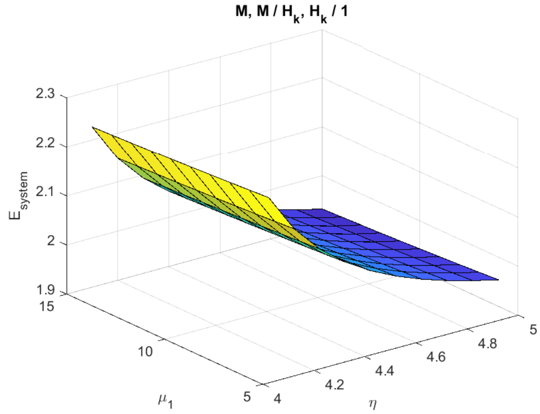

Figure 7. $M, M/H_k, H_k/1$ - η, μ_1 vs E_{system}

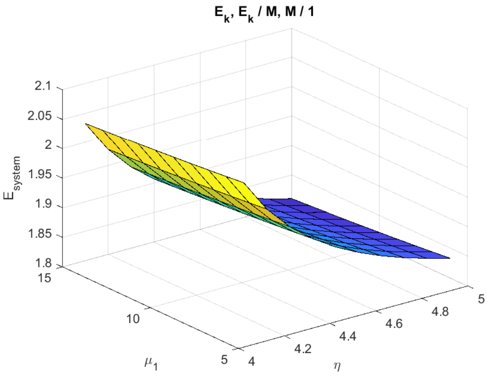

Figure 8. $E_k, E_k/M, M/1$ - η, μ_1 vs E_{system}

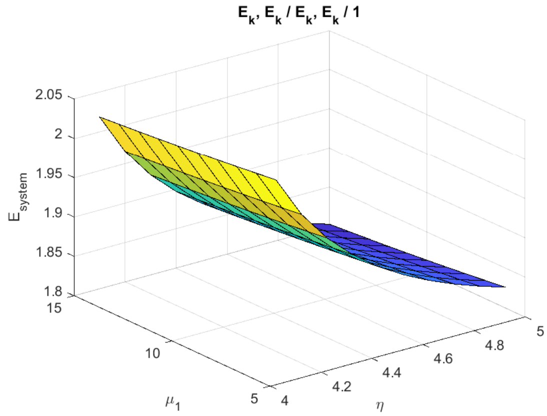

Figure 9. $ER_k, ER_k/ER_k, ER_k/1$ - η, μ_1 vs E_{system}

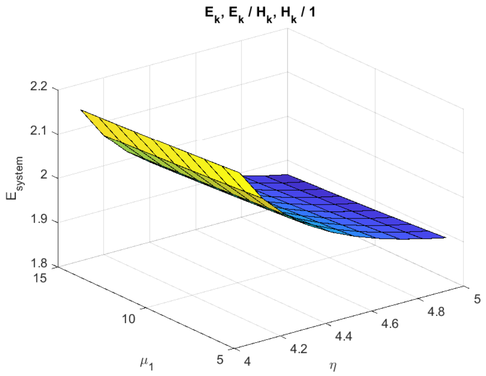

Figure 10. $E_k, E_k/H_k, H_k/1$ - η, μ_1 vs E_{system}

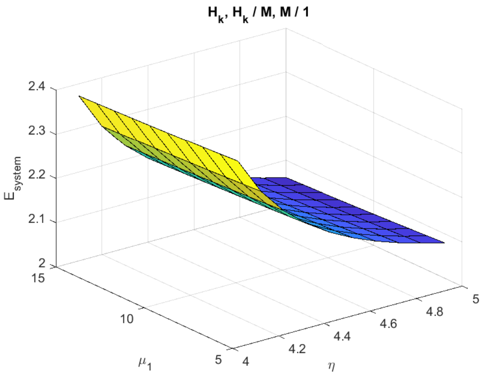

Figure 11. $H_k, H_k/M, M/1$ - η, μ_1 vs E_{system}

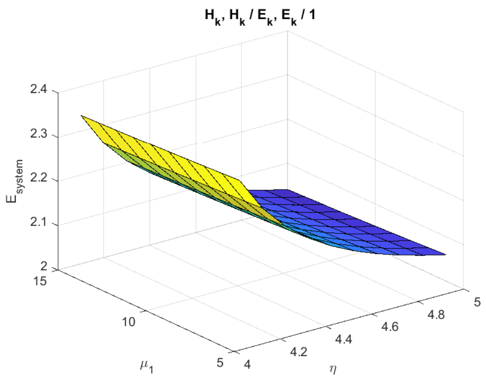

Figure 12. $H_k, H_k/E_k, E_k/1$ - η, μ_1 vs E_{system}

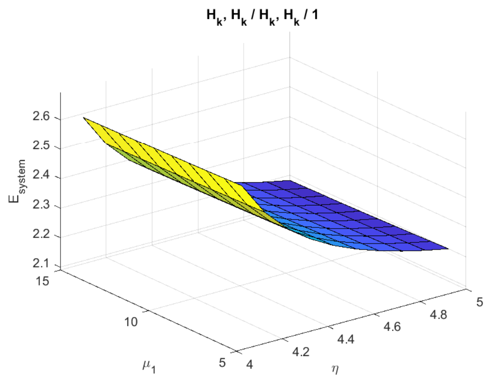

Figure 13. $H_k, H_k/H_k, H_k/1$ - η, μ_1 vs E_{system}

10. Conclusion

A single server retrial queueing inventory system with two-way communication, (s, S) replenishment policy, Bernoulli vacation, feedback, and impatient customers was investigated in this article. The stability condition, busy period of our system has been derived. Additionally, we discussed the cost analysis. The effects of several settings on a few system measures using 2D and 3D graphs are discussed. In future work it can be extended to inventory queueing models for BMAP arrivals with multi-channels.

Acknowledgments

The authors sincerely acknowledge to the Editor and the anonymous reviewers for their insightful comments, constructive suggestions, and valuable recommendations, which have greatly contributed to enhancing the quality, clarity, and overall presentation of this manuscript.

References

[1] Verma, A., Samanta, S. K., & Isotupa, K. S. (2025). A batch demands queueing-inventory model with positive replenishment time. *Queueing Models and Service Management*, 8(1), 1–20.

[2] Artalejo, J. R., & Phung-Duc, T. (2013). Single server retrial queues with two way communication. *Applied Mathematical Modelling*, 37(4), 1811–1822.

[3] Artalejo, J. R. (2010). Accessible bibliography on retrial queues: Progress in 2000–2009. *Mathematical and Computer Modelling*, 51, 1071–1081.

[4] Anbazhagan, N., Joshi, G. P., Suganya, R., Amutha, S., Vinitha, V., & Shrestha, B. (2022). Queueing-inventory system for two commodities with optional demands of customers and MAP arrivals. *Mathematics*, 10(11), 1801.

[5] Ayyappan, G., Archana Alias Gurulakshmi, G., & Somasundaram, B. (2022). Analysis of $MAP_1^I, PH_2^O/PH_1^I, PH_2^O/1$ retrial queue with two-way communication, optional service, single vacation, closedown, setup and balking. *International Journal of Mathematics in Operational Research*, 23(4), 427–455.

[6] Ayyappan, G., & Arulmozhi, N. (2024). A $MAP/PH_1, PH_2/2$ inventory queueing sytem with two commodity, multiple vacation, server feedback, working breakdown, repair and emergency replenishment. *Reliability: Theory & Applications*, 2024(19), 805–823.

[7] Ayyappan, G., & Meena, S. (2024). Analysis of $MMAP/PH(1), PH(2)/1$ preemptive priority queueing model with single vacation, repair and impatient customers. *Applications and Applied Mathematics: An International Journal (AAM)*, 19(1), 2.

[8] Ayyappan, G., & Thilagavathy, K. (2025). Analysis of (MAP, PH)/(PH, PH)/1 retrial queueing model with standby server, collision of orbital customers, breakdowns, two-way communication, phase type repairs, constant retrial rate and impatient behaviour of customers. *International Journal of Operational Research*, 52(1), 1–50.

[9] Chakravarthy, S.R. (2010). Markovian Arrival Process, Wiley Encyclopaedia of Operations Research and Management, John Wiley and Sons, Inc., USA.

[10] Shajin, D., & Melikov, A. (2025). Queueing-inventory system with return of purchased items and customer feedback. *RAIRO Opertaions Research*, 59, 1443–1473.

[11] Falin, G., & Templeton, J. G. (2023). Retrial Queues. Taylor & Francis, New York.

[12] Haghighi, A. M., & Mishev, D. P. (2007). A tandem queueing system with task-splitting, feedback, and blocking, *International Journal of Operational Research*, 2(2), 208–230.

[13] Haghighi, A. M., Chukova, S. S., & Mishev, D.P. (2011). Single-server Poisson queueing system with splitting and delayed-feedback: Part I. *International Journal of Mathematics in Operational Research*, 3(1), 1–21.

[14] Haghighi, A. M., & Mishev, D. P. (2016). Busy period of a single-server Poisson queueing system with splitting and batch delayed-feedback. *International Journal of Mathematics in Operational Research*, 8(2), 239–257.

[15] Krishnamoorthy, A., Nair, S. S., & Narayanan, V. C. (2011). An inventory model with server interruptions and retrials. *Operational Research*, 12(2), 151–171.

[16] Latouche, G., & Ramaswami, V. (1999). Introduction of Matrix Analytic Methods in Stochastic Modeling, Society for Industrial and Applied Mathematics, Philadelphia.

[17] Nazarov, A., Sztrik, J., & Kvach, A. (2018). Asymptotic Sojourn Time Analysis of Finite-Source M/M/1 Retrial Queuing System with Two-Way Communication. *Information Technologies and Mathematical Modelling. Queueing Theory and Applications*, Springer International Publishing, 172–183.

[18] Neuts, M. F. (1979). A versatile Markovian point process. *Journal of Applied Probability*, 16(4), 764–779.

[19] Neuts, M. F. (1981). Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach. The Johns Hopkins University Press, Baltimore.

[20] Neuts, M. F. (1984). Matrix-analytic methods in queuing theory. *European Journal of Operational Research*, 15(1), 2–12.

[21] Hakmi, S., Djabali, Y., Zougab, N., & Aïssani, D. (2026). The non-parametric asymmetric Kernel method in the study of M/G/1 queue with optional second service. *Queueing Models and Service Management*, 9(2), 1–14.

[22] Steeb, W. H., & Hardy, Y. (2011). Matrix Calculus and Kronecker Product: A Practical Approach to Linear and Multilinear Algebra. World Scientific Publishing, Singapore.

[23] Phung-Duc, T. (2019). *Retrial Queueing Models: A Survey on Theory and Applications*. Stochastic Operations Research in Business and Industry, Performance (cs.PF); Probability (math.PR), arXiv:1906.09560 [cs.PF]

[24] Ushakumari, P. V. (2017). A retrial inventory system with an unreliable server. *International Journal of Mathematics in Operational Research*, 10(2), 190–210.

[25] Yadavalli, V. S. S., Sivakumar, B., & Arivarignan, G. (2008). Inventory system with renewal demands at service facilities. *International Journal of Production Economics*, 114(1), 252–264.