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Study of an Early Arrival System on $\mathbf{Geo^X/G/1}$ Queue with Single Vacation

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Abstract: This paper investigates the $Geo^X/G/1$ queue with single vacation in an early arrival system. The time between arrival batches follows a geometric distribution, while the service and vacation durations are expressed as integral multiples of a slot duration and can follow arbitrary distributions. This study focuses on analyzing the system length distributions at different time epochs, the waiting time distribution for a random customer within a batch, and performing a cost analysis using the theory of difference equations and the supplementary variable technique. This method has an advantage over traditional queueing analysis techniques, as it eliminates the need to compute the transition probability matrix for the embedded system length process. We evaluate key performance metrics and demonstrate the computational process through numerical examples.

Keywords: Cost analysis, difference equation, early arrival system, queueing, single vacation, waiting time distribution.

1. Introduction

Queueing theory is a mathematical approach used to study the behavior of waiting lines or queues in various systems. Discrete-time queues are essential for studying systems where events occur at specific intervals, whereas continuous-time queues allow events to occur at any time. The discrete-time models are particularly useful in modern digital and cyber-physical systems, where operations are naturally segmented into time slots. These models are essential for modern real-world applications, such as in telecommunications systems, where they are used to provide efficient and smooth data transfer by optimizing data packet scheduling in 5G networks and internet routers. Computer networks depend on this to minimize delays in multimedia services (such as streaming data, music, and video) and to dynamically allocate bandwidth during periods of high usage. Queueing models are also used in service systems, such as call centers and hospitals, to balance staff availability with customer demand for reducing wait times and improving efficiency. These models are very helpful to digital communication systems because they divide time into intervals, which makes it

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possible to handle data packets efficiently and assist in controlling traffic. Power savings in mobile communications are becoming more and more crucial due to the quick rise in data traffic and the small battery capacity of mobile devices. User equipment (UE) uses a lot more power as communication services become more complicated, and battery technology has not kept up with the rate of improvements. One effective strategy to address this challenge is enabling sleep modes for UEs, allowing devices to enter low-power states when idle. This idea is closely related to vacation models in queueing theory, which have been widely used to study the performance of wireless networks under varying traffic conditions. More information about it can be found in Jung et al. [13] and Jayadi et al. [11]. To better understand traffic behavior and improve system efficiency, researchers utilize discrete-time queues to evaluate scenarios involving simultaneous arrivals and departures.

In most queueing models, the server remains idle when it has finished servicing the existing customers in the system and waits for the next customers to arrive. However, once the service is completed and the queue is empty, the server may temporarily leave the service area to focus on another task, which is called the server's vacation. When the server returns from a vacation and observes one or more customers waiting, it serves them until the system is empty, after which it departs for another vacation. When the server returns and finds no customers waiting, it stays idle until one arrives; on the other hand, it immediately begins another vacation and continues this cycle until a customer arrives. Queueing theory describes the first scenario as a single vacation policy and the second as a multiple vacation policy. Classical queueing theory assumes that the server is always available to serve customers. However, this is only sometimes the case in real-life scenarios. A more realistic scenario is when the server has to take breaks for various reasons, such as maintenance, repairs, or controlling traffic signals. Researchers can design and operate more efficient and effective systems in various application domains by analyzing and understanding the behavior of queueing systems with vacations. An early arrival system (EAS) in a discrete-time queue is necessary to investigate the systems so that packets (information) are sent in the same slot whenever they arrive.

The subject of vacation queues has appeared in various forms in the literature over the past few decades. A finite buffer $M/G/1$ type queueing system with exhaustive service under both single and multiple vacation policies was studied by Kempa [16]. Many researchers became interested in queueing models with vacation phenomena, including Tian and Zhang [28], Madan and Choudhury [21], Lv et al. [20], Karan et al. [15], and Nandy and Pradhan [23], as well as the references therein. Takagi [26] examined the model using the embedded Markov chain technique and analyzed finite and infinite buffer queues (including batch arrivals) with various vacation policies. Wan and Lan [30] examined the reliability of a repairable $M/G/1$ queueing system with patient servers and vacations. Vacation in the $M/G/1$ model was explored by Afanasyev [2]. Joseph[12] did a comparative analysis of queueing systems with different impatience and activation lengths under the N policy. Vijayashree and Ambika [29] studied an $M/M/1$ queue with differentiated vacations and provided time-dependent mean and variance using Laplace transforms and continued fractions. Gray et al. [10] examined a vacation queueing model with service breakdowns using the matrix-geometric method. Madhu and Praveen [22] examine an $M/E_k/1$ queueing model with state-dependent server breakdowns and working vacations in which steady-state probabilities are derived using Chapman-Kolmogorov equations, and the average queue length is obtained through probability generating function. Chakravarthy et al. [5] analyzed a queueing model with server breakdowns, repairs, vacations, and backup. Qingqing [24] analyzed the $M^X/M/1$ queue with a two-stage vacation policy using the matrix-analytic method. Kalita et al. [14] analyzed the single-server queue with a modified vacation policy. The decomposition property of an $M^X/G/1$ queue with vacations was examined by Kleiner et al. [17]. They introduced a queueing system that alternated between working and vacation modes. Adan et al. [1] examined the synchronized reneging in queueing systems with vacations, where the vacation period follows a general distribution. Altman and Yechiali [3] examined a model with an exponentially distributed vacation period and customer reneging during the vacations. An $M^X/G/1$ queue with randomized working vacations and at most J vacations was investigated by Gao and Yao [8]. Samanta and Parveen [25] used the supplementary variable technique to study the $Geo^X/G/1$ queue under EAS setup without vacation. The discrete-time $Geo/G/1$ queue with multiple adaptive vacations was studied by Zhang and Tian [33] under the presumption that the server takes a maximum random number of vacations after servicing customers in the system. Fiems and Bruneel [7] considered the discrete-time $GI/G/1$ queue with timed vacations. Wang et al. [31] examined the discrete-time $Geo/G/1$ queue with randomized vacations and at most J vacations. Tang et al. [27] examined the reliability metrics of a discrete-time $Geo^X/G/1$ queueing system under the LAS-DA setup with multiple adaptive delayed vacations and unreliable service stations. Lan and Tang [18] considered the optimal control technique and departure process structure for a discrete-time $Geo/G/1$ queue with multiple server vacations under LAS-DA. A simple mean value analysis is used by Li et al. [19] to investigate customer joining strategies in a discrete-time $Geo/G/1$ queue with server vacations. The queueing system for processing service items under vacation and the N policy with impatient customers was investigated by Divya et al. [6]. Wang at el. [32] examined the discrete-time $Geo/G/1$ queue with disastrous and non-disastrous failures using supplementary variable technique. Using the embedded Markov chain technique, Gao and Yin [9] investigated the discrete-time $Geo^X/G/1$ queue with geometrically working vacations and vacation interruption. In contrast to the above literature, the development of the $Geo^X/G/1$ queue with vacation in discrete-time under EAS setup is noteworthy. However, there is a lack of studies on the $Geo^X/G/1$ queue with a single vacation under the EAS setup.

In this paper, our work fills the gap in the existing literature by investigating the $Geo^X/G/1$ queue with single vacation under the EAS setup. A lot of study is done under the presumption of the LAS-DA. Despite its potential utility in various contexts, such as data scheduling, telecommunications and networks, the study of the EAS setup is necessary. In this model, the server's vacation time and the customers' service times follow general distributions, and customers arrive in batches based on a batch Bernoulli process. We propose an alternative approach in this work that eliminates the need to create a transition probability matrix to determine the distributions of system length at various time epochs and the waiting time for an arbitrary customer of an arrival batch. To achieve this, we use the supplementary variable approach to mathematically build the model and apply the theory of difference equations to derive the generating functions that characterize the system's behavior. First, we calculate the probability generating functions of the system's length distribution at various time epochs. Next, we use the partial fraction method on the probability generating functions to get the probabilities in terms of the zeros of the characteristic polynomial. This method is a simple but effective way to get the probabilities. Several numerical outcomes are presented in tables and graph using the analytical findings derived from this study.

The model discussed in this paper can be applied to electric vehicle (EV) charging stations by modeling the arrival of EVs and their charging times. A geometric distribution is used to illustrate the arrival process that estimates the probability of vehicles arriving at the station at specific time intervals. A general service time distribution captures the varying service time based on factors such as battery capacity and charging power. During off-peak hours or when no vehicles are present, the station may take a vacation or temporarily shut down. To reduce waiting times and effectively manage resources, the model analyzes the behavior of the queue and helps optimize station operations, including maintenance scheduling and rate adjustments.

The remaining portions of the paper are arranged as follows. The model is addressed in Section 2. The system length distributions at different time epochs are examined in Section 3. The distribution of waiting time for any customer is found in Section 4. The numerical outcomes are shown in Section 5. Section 6 concludes the paper. Finally, we explain the way to extract the unknown probabilities from the probability generating functions in Appendix A.

2. Model description

We consider a discrete-time $Geo^X/G/1$ queueing model with single vacation in which customers arrive in batches according to batch Bernoulli process. The service time of customers and the vacation time of the server follow general distributions. We explain below the various processes involved in the model:

• **Arrival process**: Customers arrive at the system in batches according to batch Bernoulli process with rate λ, $0 < \lambda < 1$. The number of customers in each batch is determined by a random variable X which has a probability mass function (p.m.f.) $P(X = i) = g_i$, $i \geqslant 1$, and an associated probability generating function (p.g.f.) $G(z) = \sum_{k=1}^{\infty} g_k z^k$, $|z| \leqslant 1$. The mean number of customers per batch is also determined by $\overline{g} = \sum_{k=1}^{\infty} k g_k < \infty$.

• **Service process**: The service time does not always follow a particular probability distribution in real-world scenarios. Therefore, we consider here a general distribution for the service time, which can cover a wide range of probability distributions. The variables s_n, $n \geqslant 1$, show the probability that the service time is of length n slots with p.g.f. $S(z) = \sum_{n=1}^{\infty} s_n z^n$, $|z| \leqslant 1$. Let $E[S] = S^{(1)}(1)$ be the mean service time, where $f^{(r)}(\zeta)$ means the r-th order derivative of $f(z)$ calculated at $z = \zeta$.

• **Vacation process**: We refer to it as a vacation when the server remains inactive in the parent queue. The concerned model is analyzed for a single vacation. In a queueing system with single vacation, the server takes an arbitrary length of vacation when the parent queue is empty. When the server returns from a vacation, if there is at least one customer in queue, it will serve them one at a time until the queue is empty. If no customers wait for service when the server returns from a vacation, it stays idle until at least one arrives. Let v_n, $n \geqslant 1$, represent the probability that the duration of the vacation is n slots with p.g.f. $V(z) = \sum_{n=1}^{\infty} v_n z^n$, $|z| \leqslant 1$. The mean vacation time is denoted by $E[V] = V^{(1)}(1)$.

• **Early arrival system:** The time axis is divided into equal-length slots in a discrete-time queueing system. We define the time axis as $0, 1, 2, \dots, t, \dots$, and the length of each slot as unity. Since we are discussing here the early arrival system (EAS), batch arrival happens just after the beginning of a slot, and departure happens just before the end of a slot. The vacation initiation epoch takes place just after an arrival epoch and vacation termination epoch takes place just after a departure epoch. Figure 1 shows the occurrences of events at different time epochs.

• **Traffic intensity**: For system stability, the traffic intensity is $\rho = \lambda \bar{g} E[S] < 1$.

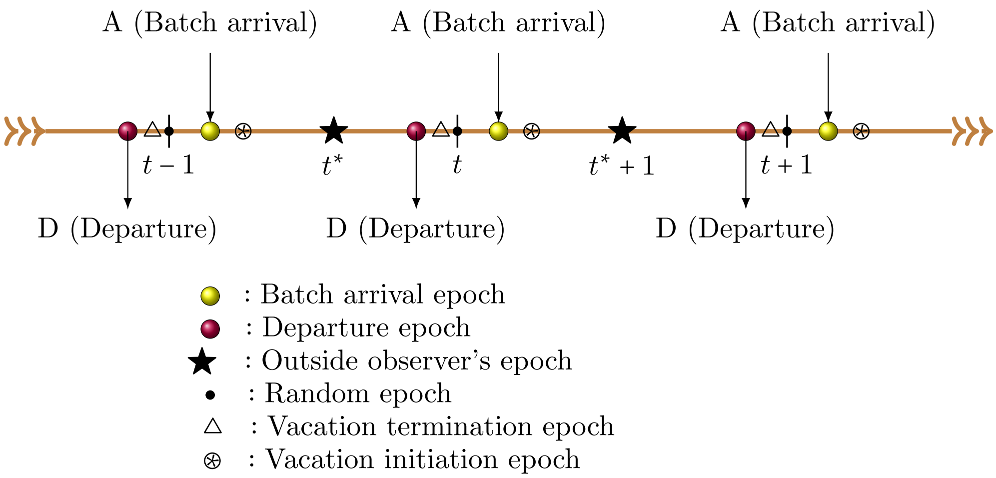

Figure 1. Significant time epochs in EAS.

3. System length distributions

The distributions of the system length at three different time epochs (outside observer's, post-departure and random) are derived in this section. We do this using the server's remaining vacation time and a customer's remaining service time as supplementary variables.

3.1. System length at outside observer's epoch

The following random variables are used to define the situation of the system at outside observer's epoch t^*:
M_{t^*}: number of customers in the system (including one in service, if any) at t^*,
U_{t^*}: remaining service time of a customer in service at t^*,
V_{t^*} : remaining vacation time of the server at t^*,
ξ_{t^*} : state of the server at time t^* and it is defined as

$$\xi_{t^*} = \begin{cases} 0, & \text{if the server is in the idle state,} \\ 1, & \text{if the server is on vacation state,} \\ 2, & \text{if the server is in busy state.} \end{cases}$$

In the steady-state, let us define their joint probabilities as

$$\begin{aligned} p^o(n,u) &= \lim_{t^*\to\infty} P\{M_{t^*}=n, U_{t^*}=u, \xi_{t^*}=2\}, \quad u \geqslant 0, \ n \geqslant 1, \\ \omega^o(n,u) &= \lim_{t^*\to\infty} P\{M_{t^*}=n, V_{t^*}=u, \xi_{t^*}=1\}, \quad u \geqslant 0, \ n \geqslant 0, \\ \nu^o &= \lim_{t^*\to\infty} P\{M_{t^*}=0, \xi_{t^*}=0\}. \end{aligned}$$

We construct the following steady-state difference equations by observing the states of the system at two consecutive outside observer's epochs t^* and t^*+1, and using the remaining service and vacation times as the supplementary variables:

$$\nu^o = \nu^o(1-\lambda) + \omega^o(0,0)(1-\lambda), \tag{1}$$

$$\begin{aligned} p^o(n,u-1) &= \nu^o \lambda g_n s_u + p^o(n,u)(1-\lambda) + \sum_{i=1}^{n-1} p^o(i,u)\lambda g_{n-i} \\ &+ \sum_{i=1}^{n} p^o(i,0)\lambda g_{n+1-i} s_u + \left(p^o(n+1,0) + \omega^o(n,0)\right)(1-\lambda)s_u \\ &+ \sum_{i=0}^{n-1} \omega^o(i,0)\lambda g_{n-i} s_u, \ n \geqslant 1, u \geqslant 1, \end{aligned} \tag{2}$$

$$\omega^o(0,u-1) = \omega^o(0,u)(1-\lambda) + p^o(1,0)(1-\lambda)v_u, \quad u \geqslant 1, \tag{3}$$

$$\omega^o(n,u-1) = \omega^o(n,u)(1-\lambda) + \sum_{i=0}^{n-1} \omega^o(i,u)\lambda g_{n-i}, \quad n \geqslant 1, \quad u \geqslant 1, \tag{4}$$

where $\sum_{i=k_1}^{k_2} = 0$ if $k_2 < k_1$.
Now, using $p^{o*}(n,z) = \sum_{u=0}^{\infty} p^o(n,u)z^u$, $n \geqslant 1$ and $\omega^{o*}(n,z) = \sum_{u=0}^{\infty} \omega^o(n,u)z^u$, $n \geqslant 0$, $|z| \leqslant 1$ in (2) - (4), we obtain

$$zp^{o*}(n,z) = \nu^o \lambda g_n S(z) + (p^{o*}(n,z) - p^o(n,0))(1-\lambda) + p^o(n+1,0)(1-\lambda)S(z)$$

$$+\sum_{i=1}^{n-1}(p^{o*}(i,z)-p^{o}(i,0))\lambda g_{n-i}+\sum_{i=1}^{n}p^{o}(i,0)\lambda g_{n+1-i}S(z)$$

$$+\omega^{o}(n,0)(1-\lambda)S(z)+\sum_{i=0}^{n-1}\omega^{o}(i,0)\lambda g_{n-i}S(z),\quad n\geqslant 1, \tag{5}$$

$$z\omega^{o*}(0,z) = (1-\lambda)(\omega^{o*}(0,z)-\omega^{o}(0,0))+p^{o}(1,0)(1-\lambda)V(z), \tag{6}$$

$$z\omega^{o*}(n,z) = (1-\lambda)(\omega^{o*}(n,z)-\omega^{o}(n,0))+\sum_{i=0}^{n-1}(\omega^{o*}(i,z)-\omega^{o}(i,0))\lambda g_{n-i}, n\geqslant 1 \tag{7}$$

Using $\tilde{\omega}^{o*}(\theta,z)=\sum_{n=0}^{\infty}\omega^{o*}(n,z)\theta^{n}$ and $\tilde{\omega}^{o}(\theta,0)=\sum_{n=0}^{\infty}\omega^{o}(n,0)\theta^{n}$, $|\theta|\leqslant 1$ in (6) and (7), we obtain

$$[z-(1-\lambda+\lambda G(\theta)]\tilde{\omega}^{o*}(\theta,z)=p^{o}(1,0)(1-\lambda)V(z)-(1-\lambda+\lambda G(\theta))\tilde{\omega}^{o}(\theta,0). \tag{8}$$

Substitute $z=1-\lambda+\lambda G(\theta)$ in (8), we get

$$\tilde{\omega}^{o}(\theta,0) = \frac{p^{o}(1,0)(1-\lambda)V(1-\lambda+\lambda G(\theta))}{1-\lambda+\lambda G(\theta)}. \tag{9}$$

Put $\theta=0$ in (9), we obtain

$$\omega^{o}(0,0)=p^{o}(1,0)V(1-\lambda). \tag{10}$$

Using (10) in (1), we get

$$\nu^{o}=\frac{p^{o}(1,0)(1-\lambda)V(1-\lambda)}{\lambda}. \tag{11}$$

Using $\tilde{p}^{o*}(\theta,z)=\sum_{n=1}^{\infty}p^{o*}(n,z)\theta^{n}$ and $\tilde{p}^{o}(\theta,0)=\sum_{n=1}^{\infty}p^{o}(n,0)\theta^{n}$ in (5), we get

$$\begin{aligned}[z-(1-\lambda+\lambda G(\theta)]\tilde{p}^{o*}(\theta,z)=&\left(\frac{S(z)}{\theta}-1\right)(1-\lambda+\lambda G(\theta))\tilde{p}^{o}(\theta,0)\\ &-\omega^{o}(0,0)(1-\lambda)S(z)-p^{o}(1,0)(1-\lambda)S(z)\\ &+(1-\lambda+\lambda G(\theta))\tilde{\omega}^{o}(\theta,0)S(z)+\nu^{o}\lambda G(\theta)S(z).\end{aligned} \tag{12}$$

Substitute $z=1-\lambda+\lambda G(\theta)$ in (12), and using (9) - (11), we get

$$\tilde{p}^{o}(\theta,0)=\frac{p^{o}(1,0)\theta(1-\lambda)\{V(1-\lambda)(G(\theta)-1)+V(1-\lambda+\lambda G(\theta))-1\}S(1-\lambda+\lambda G(\theta))}{(1-\lambda+\lambda G(\theta))(\theta-S(1-\lambda+\lambda G(\theta)))}. \tag{13}$$

Let $p_{n}^{o}\equiv p^{o*}(n,1)=\sum_{u=0}^{\infty}p^{o}(n,u)$, $n\geqslant 1$, and $\omega_{n}^{o}\equiv\omega^{o*}(n,1)=\sum_{u=0}^{\infty}\omega^{o}(n,u)$, $n\geqslant 0$, represent the probability that n customers are in the system at outside observer's epoch, when the server is busy and on vacation, respectively.
Define $\omega^{o}(\theta)=\sum_{n=0}^{\infty}\omega_{n}^{o}\theta^{n}$ and substitute $z=1$ in (8) and using (9), we get

$$\omega^{o}(\theta) = \frac{p^{o}(1,0)(1-\lambda)(1-V(1-\lambda+\lambda G(\theta)))}{\lambda(1-G(\theta))}. \tag{14}$$

Define $p^o(\theta) = \sum_{n=1}^{\infty} p_n^o \theta^n$ and substitute $z = 1$ in (12), then using (9) - (11) and (13), we obtain

$$p^o(\theta) = \frac{p^o(1,0)\theta(1-\lambda)\{V(1-\lambda)(G(\theta)-1) + V(1-\lambda+\lambda G(\theta)) - 1\}(1 - S(1-\lambda+\lambda G(\theta)))}{\lambda(1-G(\theta))(\theta - S(1-\lambda+\lambda G(\theta)))}. \quad (15)$$

Let $\Pi^o(\theta) = \nu^o + p^o(\theta) + \omega^o(\theta)$. Using the normalizing condition $\Pi^o(1) = 1$, we get

$$p^o(1,0) = \frac{\lambda(1-\rho)}{(1-\lambda)(V(1-\lambda) + \lambda E[V])}. \quad (16)$$

Now, (11), (14) and (15) are completely known to us. The extraction of the probabilities p_n^o, $n \geqslant 1$ and ω_n^o, $n \geqslant 0$, are explained in Appendix A.

Remark 1: By setting $V(z) = z$, i.e., when the server never takes vacation but remains idle, our model reduces to the standard non-vacation $Geo^X/G/1$ queueing system. With this assumption, $\Pi^o(\theta) = \nu^o + p^o(\theta) + \omega^o(\theta)$ simplifies to

$$\Pi^o(\theta) = \frac{(1-\rho)(1-\theta)S(1-\lambda+\lambda G(\theta))}{S(1-\lambda+\lambda G(\theta)) - \theta}.$$

This simplified result is identical to Samanta and Parveen [25, Eq. 9]. It also matches the expressions given in Takagi [26, Eq. 1.61] and Bruneel and Kim [4, Eq. 1.21] for the $Geo^X/G/1$ queue with late arrival system with delayed access as it should be.

3.2. System length at post-departure epoch

Let p_k^+ and ω_k^+, $k \geqslant 0$, denote the probability that k customers are in the system at service completion and vacation termination epochs, respectively. Using the probabilistic argument, p_k^+ and ω_k^+ can be found by constructing a connection between the post-departure and outside observer's epochs as

$$\begin{aligned} p_k^+ &= \lim_{t^* \to \infty} \frac{Pr[M_{t^*} = k+1,\ U_{t^*} = 0,\ \xi_{t^*} = 2]}{E^*}, \\ &= \frac{1}{E^*} p^o(k+1, 0), \quad k \geqslant 0, \quad (17) \\ \omega_k^+ &= \lim_{t^* \to \infty} \frac{Pr[M_{t^*} = k+1,\ V_{t^*} = 0,\ \xi_{t^*} = 1]}{E^*}, \\ &= \frac{1}{E^*} \omega^o(k, 0), \quad k \geqslant 0, \quad (18) \end{aligned}$$

where E^* denotes the mean inter-departure time.
Using $p^+(\theta) = \sum_{n=0}^{\infty} p_n^+ \theta^n$ and $\omega^+(\theta) = \sum_{n=0}^{\infty} \omega_n^+ \theta^n$ in (17) and (18), we get

$$\begin{aligned} \tilde{p}^o(\theta, 0) &= \theta E^* p^+(\theta), \quad (19) \\ \tilde{\omega}^o(\theta, 0) &= E^* \omega^+(\theta). \quad (20) \end{aligned}$$

Using (17) for $n = 0$ and (19) in (13), we get

$$p^+(\theta) = \frac{p_0^+(1-\lambda)\{V(1-\lambda)(G(\theta)-1) + V(1-\lambda+\lambda G(\theta)) - 1\}S(1-\lambda+\lambda G(\theta))}{(1-\lambda+\lambda G(\theta))(\theta - S(1-\lambda+\lambda G(\theta)))}. \quad (21)$$

Using (17) for $n = 0$ and (20) in (9), we get

$$\omega^+(\theta) = \frac{p_0^+(1-\lambda)V(1-\lambda+\lambda G(\theta))}{1-\lambda+\lambda G(\theta)}. \quad (22)$$

Using the normalizing condition $p^+(1) + \omega^+(1) = 1$, we get

$$p_0^+ = \frac{1-\rho}{(1-\lambda)[1-\rho+\lambda\overline{g}E[V] + \overline{g}V(1-\lambda)]}.$$

Now, both the rational functions (21) and (22) are completely known to us. The extraction of the probabilities p_n^+, $n \geqslant 1$ and ω_n^+, $n \geqslant 0$, are explained in Appendix A.

Remark 2: By setting $V(z) = z$, i.e., when the server never takes vacation but remains idle, our model reduces to the standard non-vacation $Geo^X/G/1$ queueing system. With this assumption, equation (21) simplifies to

$$p^+(\theta) = \frac{(1-\rho)(1-G(\theta))S(1-\lambda+\lambda G(\theta))}{\overline{g}(1-\lambda+\lambda G(\theta))(S(1-\lambda+\lambda G(\theta)) - \theta)}.$$

This simplified result is same as Samanta and Parveen [25, Eq. 19]. It also matches with the result of Takagi [26, Eq. 1.117] for the $Geo^X/G/1$ queue in EAS which is derived using the embedded Markov chain technique.

3.3. System length at random epoch

The following random variables define the state of the system at a random epoch t:
Y_t: number of customers in the system (including one in service, if any) at t,
I_t: remaining service time of a customer in service at t,
J_t: remaining vacation time of the server at t,
ϕ_t : state of the server at time t and it is defined as

$$\phi_t = \begin{cases} 0, & \text{if the server is in idle state,} \\ 1, & \text{if the server is on vacation state,} \\ 2, & \text{if the server is in busy state.} \end{cases}$$

In the steady-state, let us define their joint probabilities as

$$\begin{aligned} p(n,u) &= \lim_{t\to\infty} P\{Y_t = n, I_t = u, \phi_t = 2\}, \quad u \geqslant 0, \ \ n \geqslant 1, \\ \omega(n,u) &= \lim_{t\to\infty} P\{Y_t = n, J_t = u, \phi_t = 1\}, \quad u \geqslant 0, \ \ n \geqslant 0, \end{aligned}$$

$$\nu = \lim_{t\to\infty} P\{Y_t = 0, \phi_t = 0\}.$$

Observing the system's states at random epoch t and outside observer's epochs t^*, we obtain

$$\nu = \nu^o + \omega^o(0,0) + p^o(1,0), \tag{23}$$
$$p(n,u-1) = p^o(n,u) + p^o(n+1,0)s_u + \omega^o(n,0)s_u, \quad u \geqslant 1, n \geqslant 1, \tag{24}$$
$$\omega(n,u-1) = \omega^o(n,u), \quad u \geqslant 1, n \geqslant 0. \tag{25}$$

Define $p^*(n,z) = \sum_{u=0}^{\infty} p(n,u)z^u$, $n \geqslant 1$ and $\omega^*(n,z) = \sum_{u=0}^{\infty} \omega(n,u)z^u$, $n \geqslant 0$, $|z| \leqslant 1$, and use them in (24) and (25), we get

$$zp^*(n,z) = p^{o*}(n,z) - p^o(n,0) + p^o(n+1,0)S(z) + \omega^o(n,0)S(z), n \geqslant 1, \tag{26}$$
$$z\omega^*(n,z) = \omega^{o*}(n,z) - \omega^o(n,0), \quad n \geqslant 0. \tag{27}$$

Let $p_n \equiv p^*(n,1) = \sum_{u=0}^{\infty} p(n,u)$, $n \geqslant 1$, and $\omega_n \equiv \omega^*(n,1) = \sum_{u=0}^{\infty} \omega(n,u)$, $n \geqslant 0$, represent the probability that n customers are in the system at random epoch, when the server is busy and on vacation, respectively. Substituting $z = 1$ in (26) and (27) then using (17) and (18), we obtain

$$p_n = p_n^o + E^*(p_n^+ - p_{n-1}^+ + \omega_n^+), n \geqslant 1,$$
$$\omega_n = \omega_n^o - E^*\omega_n^+, \quad n \geqslant 0,$$

where E^* can be obtained by using (16) in (17) for $k = 0$ as

$$E^* = \frac{\lambda(1-\rho)}{p_0^+(1-\lambda)[\lambda E[V] + V(1-\lambda)]}.$$

Remark 3: By setting $\omega^o(n,0) = 0, n \geqslant 0$ in (23) and (24), i.e., when the server never takes vacation but remains idle, our model reduces to the standard non-vacation $Geo^X/G/1$ queueing system. With this assumption, equations (23) and (24) match those given in Samanta and Parveen [25, Eqs. 22, 23].

4. Waiting time distribution

This section presents the waiting time distribution of an arbitrary customer within a batch. The probability that a customer will wait in the queue l time slots is represented by $\varpi(l)$, $l \geqslant 0$. Further, let $W(\theta) = \sum_{l=0}^{\infty} \varpi(l)\theta^l$. The mean waiting time in the queue (W_q) is the first-order derivative of $W(\theta)$ evaluated at $\theta = 1$. Note that an arbitrary customer of a batch could observe the system upon arrival in any one of the following scenarios.

Case 1: Let $\varpi_1(l)$, $l \geqslant 0$, be the steady-state probability that, upon arrival of the batch, any arbitrary customer would wait in the queue for l time slots and observe that the server is in idle state. Further, let $W_1(\theta) = \sum_{l=0}^{\infty} \varpi_1(l)\theta^l$. An arbitrary customer in a batch with position

r must wait in queue for the services of $(r-1)$ customers to be completed. Consequently, we have

$$W_1(\theta) = \nu \sum_{r=1}^{\infty} g_r^- [S(\theta)]^{r-1}, \tag{28}$$

where $g_r^- = \frac{1}{\bar{g}} \sum_{j=r}^{\infty} g_j$, $r \geqslant 1$, signifies the probability that the position of an arbitrary customer in an arriving batch is r.
Using (23) in (28), after simplification, we obtain

$$W_1(\theta) = (\nu^o + \omega^o(0,0) + p^o(1,0)) \left(\frac{1 - G(S(\theta))}{\bar{g}(1 - S(\theta))} \right).$$

Case 2: Let $\varpi_2(l)$, $l \geqslant 0$, be the probability that, at the moment of batch arrival, an arbitrary customer would see the server in a busy state and wait in the queue for l time slots. Moreover, let $W_2(\theta) = \sum_{l=0}^{\infty} \varpi_2(l)\theta^l$. An arbitrary customer of a batch whose position is r must wait in queue for the service completions of the customer currently being serviced and the $(n+r-2)$ customers in front of the arbitrary customer. Consequently, we get

$$W_2(\theta) = \sum_{n=1}^{\infty} \theta p^*(n,\theta) \sum_{r=1}^{\infty} g_r^- [S(\theta)]^{n+r-2}. \tag{29}$$

After simplification of (29), we obtain

$$W_2(\theta) = \left(\frac{\theta \tilde{p}^*(S(\theta),\theta)}{S(\theta)} \right) \left(\frac{1 - G(S(\theta))}{\bar{g}(1 - S(\theta))} \right).$$

Case 3: Let $\varpi_3(l)$, $l \geqslant 0$, be the steady-state probability that, upon arrival of the batch, an arbitrary customer would wait in the queue for l time slots and observe the server is on vacation state. Further, let $W_3(\theta) = \sum_{l=0}^{\infty} \varpi_3(l)\theta^l$. Suppose the position of an arbitrary customer of the batch is r. In that case, the customer must wait in the queue for the server's remaining vacation time and service completions of $(n+r-1)$ customers waiting in front of the arbitrary customer. Therefore, we have

$$W_3(\theta) = \sum_{n=0}^{\infty} \theta \omega^*(n,\theta) \sum_{r=1}^{\infty} g_r^- [S(\theta)]^{n+r-1}. \tag{30}$$

After simplification of (30), we obtain

$$W_3(\theta) = (\theta \tilde{\omega}^*(S(\theta),\theta)) \left(\frac{1 - G(S(\theta))}{\bar{g}(1 - S(\theta))} \right).$$

Combining the above three cases, let $W(\theta) = W_1(\theta) + W_2(\theta) + W_3(\theta)$, we obtain

$$W(\theta) = \left(\nu^o + \omega^o(0,0) + p^o(1,0) + \frac{\theta \tilde{p}^*(S(\theta),\theta)}{S(\theta)} + \theta \tilde{\omega}^*(S(\theta),\theta) \right) \left(\frac{1 - G(S(\theta))}{\bar{g}(1 - S(\theta))} \right). \tag{31}$$

To determine the simplified form of $W(\theta)$ obtained in (31), we define $\tilde{p}^*(\theta, z) = \sum_{n=1}^{\infty} p^*(n, z)\theta^n$ and $\tilde{\omega}^*(\theta, z) = \sum_{n=0}^{\infty} \omega^*(n, z)\theta^n$, $|\theta| \leqslant 1$, and use them in (26) and (27), we get

$$\begin{aligned} z\tilde{p}^*(\theta, z) &= \tilde{p}^{o*}(\theta, z) + \left(\frac{S(z)}{\theta} - 1\right)\tilde{p}^o(\theta, 0) + (\tilde{\omega}^o(\theta, 0) \\ &\quad -\omega^o(0, 0) - p^o(1, 0))S(z), \qquad (32) \\ z\tilde{\omega}^*(\theta, z) &= \tilde{\omega}^{o*}(\theta, z) - \tilde{\omega}^o(\theta, 0). \qquad (33) \end{aligned}$$

Using (32) and (33) in (31), after simplification, we obtain

$$W(\theta) = \left(\nu^o + \frac{\tilde{p}^{o*}(S(\theta), \theta)}{S(\theta)} + \tilde{\omega}^{o*}(S(\theta), \theta)\right)\left(\frac{1 - G(S(\theta))}{\overline{g}(1 - S(\theta))}\right). \qquad (34)$$

Using (8), (11) and (12) in (34), we obtain

$$W(\theta) = \left(\frac{p^o(1, 0)(1 - \lambda)\{V(1 - \lambda)(\theta - 1) + \lambda(V(\theta) - 1)\}}{\lambda[\theta - (1 - \lambda + \lambda G(S(\theta))]}\right)\left(\frac{1 - G(S(\theta))}{\overline{g}(1 - S(\theta))}\right). \qquad (35)$$

The extraction of the probabilities $\varpi(l)$, $l \geqslant 0$, are given in Appendix A.

Remark 4: By setting $V(z) = z$, i.e., when the server never takes vacation but remains idle, our model reduces to the standard non-vacation $Geo^X/G/1$ queueing system. With this assumption, equation (35) simplifies to

$$W(\theta) = \frac{(1 - \rho)(1 - \theta)(1 - G(S(\theta)))}{\overline{g}(1 - \lambda + \lambda G(S(\theta)) - \theta)(1 - S(\theta))}.$$

This simplified result is same as Samanta and Parveen [25, Eq. 33]. It also matches the expressions in Takagi [26, Eqs. 1.51, 1.52a] for the $Geo^X/G/1$ queue under late arrival system with delayed access as it should be.

5. Discussion of numerical results

In this section, we provide numerical results to demonstrate the correctness of the analytical findings presented in this paper. To properly examine the system's behavior, we analyze the model with various arrival, service and vacation time distributions. Tables 1 - 3 show the numerical results of the system length distributions at different time epochs and the waiting time distribution together with other performance measures. It is evident from all of the instances examined here that $\nu^o + \sum_{n=0}^{\infty} \omega_n^o = 1 - \rho$ is true. Furthermore, we extensively investigate the performance measures of the system such as the expected number of customers in the queue denoted as $L_q = \sum_{n=1}^{\infty}(n - 1)p_n^o + \sum_{n=0}^{\infty} n\omega_n^o$ and the expected waiting time of a customer in the queue denoted as $W_q = \frac{L_q}{\lambda\overline{g}}$. The L_q and W_q are provided at the bottom of each table. Significantly, we notice that Little's law holds in all the cases as it should. These data confirm the correctness of our analytical and numerical processes. We perform all of our computations precisely, even though we display the results to seven decimal places. Furthermore, these results can be useful to other researchers when comparing our findings with those obtained using different methodologies.

Table 1. System length and waiting time distributions, when arrival batch size distribution is geometrically distributed with p.m.f. $g_k = (1-\eta)^{k-1}\eta$, $k \geqslant 1$, $(0 < \eta < 1)$ and p.g.f. $G(z) = \frac{\eta z}{1-(1-\eta)z}$. Here, we have taken $\eta = 0.45$ which gives $\overline{g} = \frac{1}{\eta} = 2.2222222$ and $\lambda = 0.085$. The service time is chosen as arbitrarily distributed with $s_1 = 0.4$, $s_5 = 0.25$ and $s_9 = 0.35$. Thus $E[S] = 4.80$. These lead to $\rho = 0.9066666$. The vacation time is chosen as arbitrarily distributed with $v_2 = 0.2$, $v_3 = 0.3$ and $v_5 = 0.5$. Hence $E[V] = 3.8$.

n	ω_n^o	p_n^o	ω_n^+	p_n^+	ω_n	p_n	l	$\varpi(l)$
0	0.0252895		0.0304307	0.0423859	0.0233306		0	0.0354068
1	0.0015310	0.0414826	0.0033839	0.0418742	0.0013131	0.0416675	1	0.0142348
2	0.0008942	0.0406642	0.0020207	0.040926	0.0007642	0.0407332	2	0.0091880
3	0.0005215	0.0393936	0.0012031	0.0396677	0.0004441	0.0393900	3	0.0073622
4	0.0003037	0.0378904	0.0007144	0.0382145	0.0002578	0.0378428	4	0.0073336
5	0.0001766	0.0362852	0.0004231	0.0366571	0.0001494	0.0362121	5	0.0103598
6	0.0001026	0.0346544	0.0002500	0.0350597	0.0000865	0.0345677	6	0.0081527
7	0.0000595	0.0330421	0.0001474	0.0334653	0.0000501	0.0329489	7	0.0071046
8	0.0000345	0.0314726	0.0000868	0.0319013	0.0000289	0.0313774	8	0.0068033
9	0.0000199	0.0299588	0.0000510	0.0303840	0.0000167	0.0298644	9	0.0132332
10	0.0000115	0.0285069	0.0000299	0.0289226	0.0000096	0.0284148	10	0.0108139
17	0.0000002	0.0200673	0.0000006	0.0203741	0.0000002	0.0199998	50	0.0059875
20	0.0000000	0.0172580	0.0000001	0.0175222	0.0000000	0.0171999	150	0.0021295
50	0.0000000	0.0038186	0.0000000	0.0038772	0.0000000	0.0038058	250	0.0007571
150	0.0000000	0.0000250	0.0000000	0.0000254	0.0000000	0.0000249	500	0.0000571
200	0.0000000	0.0000020	0.0000000	0.0000020	0.0000000	0.0000020	800	0.0000025
235	0.0000000	0.0000003	0.0000000	0.0000003	0.0000000	0.0000003	959	0.0000004
261	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	1114	0.0000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.0289608	0.9066667	0.0387831	0.9612168	0.0264642	0.9044758		0.9999999
ν^o	0.0643725							
ν					0.0690599			
			$L_q = 17.7394568$		$W_q = 93.9147713$			

Table 2. System length and waiting time distributions, when arrival batch size distribution is geometrically distributed with p.m.f. $g_k = (1-\eta)^{k-1}\eta$, $k \geqslant 1$, $(0 < \eta < 1)$ and p.g.f. $G(z) = \frac{\eta z}{1-(1-\eta)z}$. Here, we have taken $\eta = 0.8$ which gives $\overline{g} = \frac{1}{\eta} = 1.2500000$ and $\lambda = 0.055$. We choose a negative binomial distribution with p.m.f. $s_k = \sum_{k=0}^{\infty} \binom{k+r-2}{k-1}(1-\sigma)^r \sigma^{k-1}$, $r \geqslant 1$, $(0 < \sigma < 1)$, and p.g.f. $S(z) = \left(\frac{1-\sigma}{1-\sigma z}\right)^r z$ as the service time distribution. For computation purposes, we have taken $\sigma = 0.5$ and $r = 7$. These lead to $E[S] = 8.00$ and $\rho = 0.5500000$. The vacation time is taken as arbitrarily distributed with $v_2 = 0.3$, $v_3 = 0.4$ and $v_{17} = 0.3$. Thus $E[V] = 6.9$.

n	ω_n^o	p_n^o	ω_n^+	p_n^+	ω_n	p_n	l	$\varpi(l)$
0	0.1145235		0.1879503	0.2609899	0.0591345		0	0.2685350
1	0.0249696	0.2207117	0.0337556	0.1967340	0.0150219	0.2117233	1	0.0350047
2	0.0099978	0.1399827	0.0147280	0.1258396	0.0056574	0.1234305	2	0.0320505
3	0.0037684	0.0831468	0.0062327	0.0747406	0.0019317	0.0699248	3	0.0271211
4	0.0013507	0.0475572	0.0024983	0.0426601	0.0006144	0.0388394	4	0.0291767
5	0.0004642	0.0265755	0.0009487	0.0237909	0.0001846	0.0212944	5	0.0306605
6	0.0001540	0.0146344	0.0003434	0.0130820	0.0000528	0.0115797	6	0.0314219
7	0.0000496	0.0079843	0.0001193	0.0071311	0.0000144	0.0062658	7	0.0315009
8	0.0000155	0.0043310	0.0000400	0.0038663	0.0000038	0.0033807	8	0.0310518
9	0.0000048	0.0023409	0.0000130	0.0020893	0.0000009	0.0018211	9	0.0302627
10	0.0000014	0.0012626	0.0000041	0.0011268	0.0000002	0.0009801	10	0.0293016
12	0.0000001	0.0003660	0.0000003	0.0003267	0.0000000	0.0002836	17	0.0177236
13	0.0000000	0.0001969	0.0000001	0.0001758	0.0000000	0.0001525	25	0.0106123
15	0.0000000	0.0000569	0.0000000	0.0000508	0.0000000	0.0000441	50	0.0017259
20	0.0000000	0.0000026	0.0000000	0.0000023	0.0000000	0.0000024	100	0.0000458
24	0.0000000	0.0000002	0.0000000	0.0000001	0.0000000	0.0000001	150	0.0000012
26	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	175	0.0000001
30	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	185	0.0000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.1553002	0.5500000	0.2466355	0.7533645	0.0826168	0.4903808		0.9999999
ν^o	0.2946998							
ν					0.4270023			
			$L_q = 0.8186303$		$W_q = 11.9073511$			

Table 3. System length and waiting time distributions, when arrival batch size is chosen as arbitrarily distributed with a finite maximum batch size. The values for λ and the maximum batch size are considered as 0.06 and 7, respectively. The p.m.f. of batch size distribution is taken as $g_1 = 0.40$, $g_3 = 0.25$, $g_5 = 0.15$ and $g_7 = 0.20$ with $\overline{g} = 3.30$. The service time is chosen as generally distributed with $s_1 = 0.60$ and $s_3 = 0.40$. Hence $E[S] = 1.80$. These lead to $\rho = 0.3564000$. The p.g.f. of vacation time is taken as $V(z) = \frac{z^2}{(2-z)^5}$ with $E[V] = 7$.

n	ω_n^o	p_n^o	ω_n^+	p_n^+	ω_n	p_n	l	$\varpi(l)$
0	0.2023976		0.1074201	0.1626894	0.1601411		0	0.1383371
1	0.0163387	0.0689789	0.0149034	0.1167578	0.0104761	0.0567732	1	0.0680358
2	0.0009801	0.0493013	0.0011365	0.1223418	0.0005330	0.0519451	2	0.0552042
3	0.0102605	0.0517226	0.0093794	0.0903645	0.0065708	0.0428331	3	0.0712301
4	0.0012273	0.0380317	0.0014237	0.0900397	0.0006673	0.0384640	4	0.0780976
5	0.0062185	0.0379650	0.0057102	0.0683422	0.0039722	0.0316761	5	0.0584991
6	0.0011233	0.0286877	0.0013041	0.0654345	0.0006103	0.0280569	6	0.0671638
7	0.0082816	0.0275239	0.0076008	0.0326273	0.0052917	0.0176084	7	0.0572123
8	0.0014478	0.0135166	0.0016811	0.0246643	0.0007864	0.0110454	8	0.0574671
9	0.0001541	0.0102362	0.0002047	0.0196592	0.0000736	0.0083479	9	0.0469223
10	0.0007628	0.0081271	0.0008881	0.0145798	0.0004134	0.0064782	10	0.0484481
15	0.0000746	0.0016428	0.0000993	0.0029005	0.0000356	0.0012505	15	0.0196911
25	0.0000003	0.0000598	0.0000005	0.0001043	0.0000001	0.0000436	25	0.0035045
30	0.0000000	0.0000109	0.0000000	0.0000188	0.0000000	0.0000079	50	0.0000369
39	0.0000000	0.0000004	0.0000000	0.0000008	0.0000000	0.0000003	65	0.0000024
46	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	70	0.0000009
50	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	78	0.0000002
80	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	84	0.0000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.2502246	0.3563999	0.1529280	0.8470719	0.1900665	0.3103037		0.9999999
ν^o	0.3933754							
ν					0.4996298			
			$L_q = 1.4413722$	$W_q = 7.2796576$				

We now concentrate on examining the effect of the arrival batch size variance Var$[G]$ on W_q. Three distinct values of Var$[G]$ are considered in such a way that

$$\begin{aligned} \text{Var}[G] &= 5.36: \text{ when } g_1 = 0.10, g_2 = 0.25, g_3 = 0.30, g_4 = 0.20, g_6 = 0.15, \\ \text{Var}[G] &= 7.87: \text{ when } g_1 = 0.65, g_4 = 0.05, g_6 = 0.30, \\ \text{Var}[G] &= 8.61: \text{ when } g_1 = 0.40, g_3 = 0.25, g_5 = 0.15, g_7 = 0.20. \end{aligned}$$

For this aim, the service time is chosen as generally distributed with $s_1 = 0.60$ and $s_3 = 0.40$. The p.g.f. of vacation time is taken as $V(z) = \frac{z^2}{(2-z)^5}$. In Figure 2, we observe that increasing the value of λ leads to a increase in W_q. Figure 2 also shows that the value of W_q increases as Var$[G]$ increases. This is due to the fact that the value of Var$[G]$ increases with the increase of probability of a larger arrival batch size. Thus, as Var$[G]$ increases, there are more customers in the system. This indicates that the value W_q increases as Var$[G]$ increases.

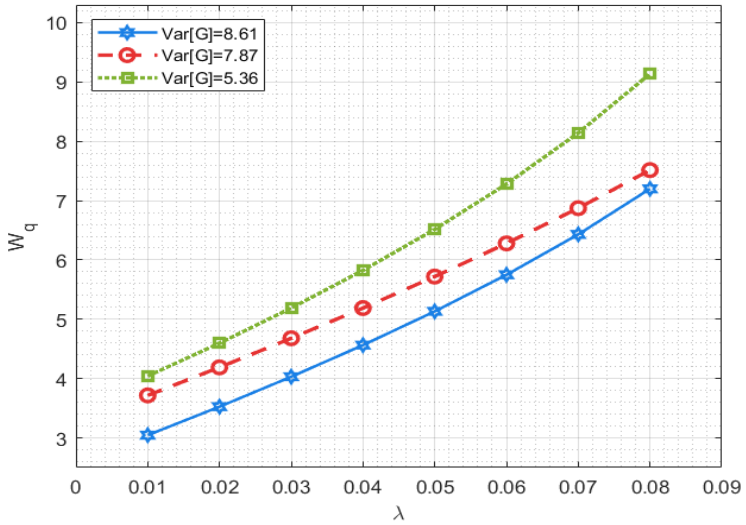

Figure 2. Effect of variance of batch arrivals

5.1. Cost analysis

Based on the previous analysis, we now construct the cost function as a linear combination of costs related to several characteristics of the system such as the number of customers in the queue, the service time, the vacation time, and the idle time of this model. This cost function is denoted as TC and it is given by

$$TC = C_1 L_q + \frac{C_2}{E[S]} + \frac{C_3}{E[V]} + \frac{C_4}{E[I]},$$

where the expected idle time is provided by

$$E[I] = \left(E[V] + \frac{1}{\lambda}\right) V(1-\lambda) + E[V]\,(1 - V(1-\lambda))$$

$$= \quad E[V] + \frac{1}{\lambda}V(1-\lambda).$$

The cost components are defined as follows:

- C_1: Holding cost per customer per unit time,
- C_2: Cost associated with service time,
- C_3: Cost associated with vacation time,
- C_4: Cost associated with idle time.

In Table 4, we consider an arrival batch size 7 with an arbitrary distribution specified as $g_1 = 0.40$, $g_3 = 0.25$, $g_5 = 0.15$ and $g_7 = 0.20$. The service time is chosen as arbitrarily distributed with $s_1 = 0.60$ and $s_3 = 0.40$, and the vacation time follows geometric distribution with p.g.f. $V(z) = \frac{\varsigma z}{1-(1-\varsigma)z}$, $(0 < \varsigma < 1)$. The parameter ς is varied as given below to achieve different expected vacation times $E[V]$. The goal is to identify the optimum value of the cost function which is indicated as bold in Table 4. We perform all of our computations precisely, even though we display the results for $E[V]$ to two decimal places.

$$\begin{aligned}
E[V] &= 2.00: \text{ when } \varsigma = 0.50\\
E[V] &= 2.50: \text{ when } \varsigma = 0.40\\
E[V] &= 3.33: \text{ when } \varsigma = 0.30\\
E[V] &= 5.00: \text{ when } \varsigma = 0.20\\
E[V] &= 10.00: \text{ when } \varsigma = 0.10\\
E[V] &= 14.28: \text{ when } \varsigma = 0.07\\
E[V] &= 16.66: \text{ when } \varsigma = 0.06\\
E[V] &= 20.00: \text{ when } \varsigma = 0.05\\
E[V] &= 24.39: \text{ when } \varsigma = 0.041.
\end{aligned}$$

In Table 5, the batch arrival is assumed to be geometrically distributed with p.g.f. $G(z) = \frac{\chi z}{1-(1-\chi)z}$, $(0 < \chi < 1)$. Here we consider $\chi = 0.45$. The service time is taken as arbitrarily distributed, and vacation time is chosen as arbitrarily distributed with p.m.f. $v_1 = 0.2$, $v_3 = 0.3$ and $v_5 = 0.5$. The values of s_1, s_3, s_5, s_6 and s_9 are adjusted to obtain different expected service times $E[S]$. The goal is to identify the optimum value of the cost function which is indicated as bold in Table 5. We perform all of our computations precisely, even though we display the results for $E[S]$ to two decimal places.

$$\begin{aligned}
E[S] &= 1.77: \text{ when } s_1 = 0.759, s_3 = 0.172, s_5 = 0.013, s_6 = 0.022, s_9 = 0.034\\
E[S] &= 2.23: \text{ when } s_1 = 0.714, s_3 = 0.132, s_5 = 0.033, s_6 = 0.042, s_9 = 0.079\\
E[S] &= 2.75: \text{ when } s_1 = 0.649, s_3 = 0.097, s_5 = 0.084, s_6 = 0.046, s_9 = 0.124\\
E[S] &= 3.33: \text{ when } s_1 = 0.569, s_3 = 0.127, s_5 = 0.050, s_6 = 0.051, s_9 = 0.203
\end{aligned}$$

Table 4. Cost analysis for varying λ and $E[V]$

$E[V]$	$\lambda = 0.03$	$\lambda = 0.04$	$\lambda = 0.05$	$\lambda = 0.06$	$\lambda = 0.07$	$\lambda = 0.08$
2.00	35.8729556	38.1016958	40.7150026	43.8156906	47.5485074	52.1238742
2.50	31.4237908	33.6912565	36.3535787	39.5131588	43.3143054	47.9669808
3.33	27.0403048	29.3959155	32.1690936	35.4608717	39.4141112	44.2372834
5.00	22.8898201	25.5050482	**28.5995239**	**32.2675456**	**36.6451898**	**41.9343339**
10.00	**20.2914712**	**24.2192993**	28.8488075	34.2127251	40.3967071	47.5632114
14.28	21.4227998	26.8793974	33.1680546	40.2438724	48.1458885	57.0112117
16.66	22.6181370	29.0047868	36.2569493	44.2915395	53.1311172	62.9079033
20.00	24.7246536	32.4559008	41.0581102	50.4092148	60.5218507	71.5304776
24.39	28.0369690	37.5567461	47.9012322	47.9012322	70.6474945	83.2146974

$$
\begin{aligned}
E[S] &= 4.00: \text{ when } s_1 = 0.507, s_3 = 0.107, s_5 = 0.044, s_6 = 0.042, s_9 = 0.300 \\
E[S] &= 4.76: \text{ when } s_1 = 0.405, s_3 = 0.103, s_5 = 0.056, s_6 = 0.051, s_9 = 0.385 \\
E[S] &= 5.66: \text{ when } s_1 = 0.241, s_3 = 0.103, s_5 = 0.062, s_6 = 0.179, s_9 = 0.415 \\
E[S] &= 6.72: \text{ when } s_1 = 0.090, s_3 = 0.092, s_5 = 0.042, s_6 = 0.277, s_9 = 0.499 \\
E[S] &= 8.00: \text{ when } s_1 = 0.005, s_3 = 0.007, s_5 = 0.092, s_6 = 0.181, s_9 = 0.715.
\end{aligned}
$$

Table 5. Cost analysis for varying λ and $E[S]$

$E[S]$	$\lambda = 0.15$	$\lambda = 0.020$	$\lambda = 0.025$	$\lambda = 0.30$	$\lambda = 0.035$	$\lambda = 0.040$
1.77	21.4045941	21.8275694	22.2760505	22.7511926	23.2542611	23.7866466
2.23	19.9102793	20.4397621	21.0104998	21.6254203	22.2878127	23.0013849
2.75	18.9340488	19.5905098	20.3101269	21.0991499	**21.9648146**	**22.9155438**
3.33	18.3121377	19.1286706	**20.0417001**	**21.0639709**	22.2107961	23.5007339
4.00	17.9587949	**18.9807604**	20.1504875	21.4935568	23.0421771	24.8374743
4.76	**17.8326364**	19.1153182	20.6233573	22.4076044	24.5360579	27.1017431
5.66	17.9200322	19.5478289	21.5255384	23.9573972	26.9946701	30.8655118
6.72	18.2622726	20.3821963	23.0752767	26.5719546	31.2473390	37.7557910
8.00	18.9408038	21.8225189	25.7254587	31.2305168	39.4666976	52.9569349

In Table 6, the batch arrival is assumed to be geometrically distributed with p.g.f. $G(z) = \frac{\Psi z}{1-(1-\Psi)z}$, $(0 < \Psi < 1)$. We consider here $\Psi = 0.8$. We define the service time as a negative binomial distribution with p.g.f. $S(z) = \left(\frac{1-\Omega}{1-\Omega z}\right)^r z$, $r \geqslant 1$ and $0 < \Omega < 1$. Here we have taken $r = 7$. The vacation time is taken as arbitrarily distributed with p.m.f. $v_2 = 0.3$, $v_3 = 0.4$ and $v_{17} = 0.3$. The value of Ω in $S(z)$ is adjusted to obtain different expected service times $E[S]$. The goal is to identify the optimum value of the cost function which is indicated as bold in Table 6. We perform all of our computations precisely, even though we display the results for $E[S]$ to two decimal places.

$$E[S] = 1.77: \text{ when } \Omega = 0.10$$

$$
\begin{aligned}
E[S] &= 2.23: \text{ when } \Omega = 0.15 \\
E[S] &= 2.75: \text{ when } \Omega = 0.20 \\
E[S] &= 3.33: \text{ when } \Omega = 0.25 \\
E[S] &= 4.00: \text{ when } \Omega = 0.30 \\
E[S] &= 4.76: \text{ when } \Omega = 0.35 \\
E[S] &= 5.66: \text{ when } \Omega = 0.40 \\
E[S] &= 6.72: \text{ when } \Omega = 0.45 \\
E[S] &= 8.00: \text{ when } \Omega = 0.50.
\end{aligned}
$$

Table 6. Cost analysis for varying λ and $E[S]$

$E[S]$	$\lambda = 0.043$	$\lambda = 0.045$	$\lambda = 0.047$	$\lambda = 0.050$	$\lambda = 0.053$	$\lambda = 0.055$
1.77	16.9861303	17.1189169	17.2544658	17.4627920	17.6768857	17.8226917
2.23	15.3697199	15.5104791	15.6543003	15.8756131	16.1034097	16.2587621
2.75	14.2554442	14.4071816	14.5624688	14.8019290	15.0490605	15.2179909
3.33	13.4859278	13.6530849	13.8245925	14.0899564	14.3649833	14.5536791
4.00	12.9830138	13.1723943	13.3674704	13.6708618	13.9873528	14.2057367
4.76	12.7173835	**12.9398892**	**13.1704377**	**13.5317781**	**13.9124209**	**14.1773344**
5.66	**12.7015117**	12.9756509	13.2621676	13.7163941	14.2018971	14.5441608
6.72	13.0017464	13.3614527	13.7422319	14.3562749	15.0271223	15.5093778
8.00	13.7835667	14.2980914	14.8534182	15.7727963	16.8123402	17.5833666

The cost optimization results presented in Tables 4 - 6 provide valuable insights for operators of EV charging stations. The bold values in these tables identify parameter combinations that minimize total costs. For instance, Table 4 demonstrates that the setting of $E[V] = 10.00$ with arrival rate $\lambda = 0.03$ achieves the lowest cost of 20.2914712. Similarly, in Table 5, the combination of $E[S] = 4.76$ and $\lambda = 0.15$ yields the lowest cost of 17.8326364. Operators should aim to set the values of $E[S]$ and $E[V]$ to optimize costs. As increasing λ beyond these optimal values can lead to raise costs.

6. Conclusion

In this paper, we have analyzed a discrete-time $Geo^X/G/1$ queueing system, where arrival occurs in batches according to the Bernoulli process. Throughout the investigation, we have assumed the EAS setup. Using a system of difference equations and the supplementary variable technique, we have obtained probability generating functions at outside observer's, post-departure, and random epochs. We have also investigated the waiting time distribution for a random customer in a batch. We have done a cost analysis and obtained several performance measures. A key novelty of this work is the avoidance of constructing the transition probability matrix, which is a common and often complex step in traditional queueing analysis. Unlike the conventional approaches that rely on Markov chain and a set of balance equations, our method offers a more efficient and straightforward analytical framework. We have included several numerical examples to make this study comprehensive and valuable to other researchers. This methodology can be extended to more complex systems, such as the $Geo^X/G^{a,b}/1$ queue with vacation or models involving correlated arrivals, priority-based service, or state-dependent behavior.

Declarations

- Competing interests: The authors report no competing interests to declare.
- Data availability: No external datasets were used in this study. All data relevant to the research are provided within the paper.
- Author contribution: These authors contributed equally to this work.

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Appendix

A. Extraction of the Probabilities from Completely Known Rational Function

We consider the function $T(\theta) = \sum_{n=1}^{\infty} T_n\theta^n$, which is used for $p^o(\theta)$ to derive p_n^o, $n \geqslant 1$, from (15). Similarly, we take into account the function $T(\theta) = \sum_{n=0}^{\infty} T_n\theta^n$ which is employed as $\omega^o(\theta)$, $p^+(\theta)$, $\omega^+(\theta)$ and $W^*(\theta)$ to produce ω_n^o, p_n^+, ω_n^+ and $\varpi(n)$, $n \geqslant 0$, from (14), (21), (22) and (35), respectively. Therefore, we discuss here to determine T_n from $T(\theta)$, which is enough to know the procedure to find all the unknown probabilities analyzed in the above sections. To do this, we use the partial fraction method, which depends on the knowledge of the zeros in the denominator of $T(\theta)$. For simplicity, we represent $T(\theta)$ in rational form as

$$T(\theta) = \frac{N(\theta)}{D(\theta)}. \tag{36}$$

Since the zeros of $D(\theta)$ with absolute values smaller than or equal to one are also the zeros of $N(\theta)$ and have no further importance in the computation of the partial fraction, we need to know the zeros of $D(\theta)$ whose absolute value is larger than one. When $\rho < 1$, Rouche's theorem can be used to demonstrate that the function $D(\theta)$ has one zero within and on the unit circle. Let $N(\theta)$ and $D(\theta)$ be the polynomials of degree b and a, respectively, after cancellation of all common zeros inside and on the unit circle. The degrees of $N(\theta)$ and $D(\theta)$ depend on the distributions of arrival batch size, service time, and vacation time.

• **When all the zeros of $D(\theta)$ in $|\theta| > 1$ are distinct**

Let α_r, $r = 1, 2, \ldots, a$, denote the zeros of $D(\theta)$ whose absolute value is greater than one. Now use the partial fraction expansion on (36), we write it as

$$\begin{aligned} T(\theta) &= \sum_{k=0}^{b-a} H_k\theta^k + \sum_{r=1}^{a} \frac{B_r}{\theta - \alpha_r}, \quad \text{for } b \geqslant a, & (37)\\ &= \sum_{r=1}^{a} \frac{B_r}{\theta - \alpha_r}, \quad \text{for } b < a, & (38)\end{aligned}$$

where

$$\begin{aligned} B_r &= \frac{N(\alpha_r)}{D^{(1)}(\alpha_r)}, \quad r = 1, 2, \ldots, a,\\ H_k &= \frac{1}{k!}\left(T^{(k)}(0) + \sum_{r=1}^{a} \frac{k!B_r}{\alpha_r^{k+1}}\right), \; k = 0, 1, \ldots, b-a,\end{aligned}$$

with

$$\begin{aligned} T^{(0)}(0) &= \frac{N(0)}{D(0)},\\ T^{(k)}(0) &= \frac{N^{(k)}(0) - \sum_{j=0}^{k-1}\binom{k}{j}T^{(j)}(0)D^{(k-j)}(0)}{D(0)}, \; k = 1, 2, \ldots, b-a.\end{aligned}$$

Collect the coefficients of θ^k from (37), we obtain

$$\begin{aligned} T_k &= H_k - \sum_{r=1}^{a} \frac{B_r}{\alpha_r^{k+1}}, \quad k = 0, 1, \ldots, b-a,\\ T_k &= -\sum_{r=1}^{a} \frac{B_r}{\alpha_r^{k+1}}, \quad k \geqslant b-a+1.\end{aligned}$$

Similarly, collect the coefficients of θ^k from (38), we obtain

$$T_k = -\sum_{r=1}^{a} \frac{B_r}{\alpha_r^{k+1}}, \quad k \geqslant 0.$$

• When some of the zeros of $D(\theta)$ in $|\theta| > 1$ are repeated

Suppose $D(\theta)$ has zeros $\beta_1, \beta_2, \ldots, \beta_l, \beta_{l+1}, \ldots, \beta_m$,($m < a$) with multiplicity $t_1, t_2, \ldots, t_l, t_{l+1}, \ldots, t_m$, where each $t_j \geqslant 1$, $j = 1, 2. \ldots, m$. Use the partial fraction expansion on (36), we write it as

$$\begin{aligned} T(\theta) &= \sum_{k=0}^{b-a} H_k \theta^k + \sum_{i=1}^{m} \sum_{h=1}^{t_i} \frac{D_{i,h}}{(\theta - \beta_i)^h}, \quad \text{for } b \geqslant a, &(39)\\ &= \sum_{i=1}^{m} \sum_{h=1}^{t_i} \frac{D_{i,h}}{(\theta - \beta_i)^h}, \quad \text{for } b < a, &(40) \end{aligned}$$

where

$$\begin{aligned} D_{i,h} &= \left[\frac{F^{(t_i-h)}(\theta) - \sum_{j=0}^{t_i-h-1} \binom{t_i-h}{j} P_i^{(j)}(\theta) Q_i^{(t_i-h-j)}(\theta)}{(t_i - h)! Q_i(\theta)} \right]_{\theta=\beta_i}, \quad h = t_1, t_2, \ldots, t_i, \\ & \qquad i = 1, 2, \ldots, m, \\ H_k &= \frac{1}{k!} \left(T^{(k)}(0) + (-1)^{k+h} \sum_{r=1}^{m} \sum_{h=1}^{t_r} \frac{k!(r + k - 1)! D_{r,h}}{(r - 1)! \beta_r^{k+h}} \right), \ k = 0, 1, \ldots, b - a, \end{aligned}$$

with

$$\begin{aligned} F(\theta) &= \frac{N(\theta)}{\Lambda}, \\ P_i(\theta) &= \sum_{h=1}^{t_i} D_{i,h} (\theta - \beta_i)^{t_i - h}, \quad i = 1, 2, \ldots, m, \\ Q_i(\theta) &= \prod_{k=1, k \neq i}^{m} (\theta - \beta_k)^{t_k}, \quad i = 1, 2, \ldots, m, \end{aligned}$$

where Λ is the highest degree coefficient of $D(\theta)$.
Collect the coefficients of θ^k from (39), we obtain

$$\begin{aligned} T_k &= H_k - \sum_{i=1}^{m} \sum_{h=1}^{t_i} D_{i,h} v(k, i, h), \quad k = 0, 1, 2, \ldots, b - a, \\ T_k &= \sum_{i=1}^{m} \sum_{h=1}^{t_i} D_{i,h} v(k, i, h), \quad k > b - a, \end{aligned}$$

where

$$\sum_{n=0}^{\infty} v(n, i, h) \theta^n = \frac{1}{(\theta - \beta_i)^h}. \tag{41}$$

Take log on both sides of (41), we obtain

$$\log\left(\sum_{n=0}^{\infty} v(n,i,h)\theta^n\right) = -h\log(\theta-\beta_i). \tag{42}$$

Upon differentiating (42) with respect to θ, we get

$$(\theta-\beta_i)\sum_{n=1}^{\infty} nv(n,i,h)\theta^{n-1} \;=\; -h\sum_{n=0}^{\infty} v(n,i,h)\theta^n. \tag{43}$$

Collecting the constant terms from (41) and coefficients of θ^n, $n \geqslant 1$, from (43), we obtain

$$\begin{aligned} v(0,i,h) &= \frac{1}{(-\beta_i)^h}, \\ v(n,i,h) &= \frac{(n-1+h)v(n-1,i,h)}{n\beta_i}. \end{aligned}$$

which leads to

$$v(n,i,h) = \frac{(h+n-1)!(-1)^h}{n!(h-1)!\beta_i^{h+n}}, \quad n \geqslant 0.$$

Similarly, collecting the coefficients of θ^n from both the sides of (40), we obtain

$$T_n = \sum_{i=1}^{m}\sum_{h=1}^{t_i} D_{i,h}v(n,i,h), \quad n \geqslant 0.$$