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A Time-dependent Tandem BMAP with Balking and Batch Service with Possible Breakdown and Delayed Service

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Abstract: In this paper, a time-dependent queueing system containing two stages in tandem has been considered. Carriers carry jobs by bulks of various sizes and arrive at the first stage according to a Poisson distribution. There is a possibility of balking and, hence, jobs attend with some probability at an infinite size buffer, located at the entrance of the first stage. As the jobs attend, they will be placed randomly in the buffer with some type of identification for later to be served based on that order, that is, first-come – first-served rule. For jobs to be served by batches, they will be grouped with a minimum and a maximum limit and will be moved to be served by a single server. However, before being served, jobs must go through a procedure that causes service be performed with delay. There is also a possibility of a server breakdown that would require to be repaired, which will affect arrivals, that cause another possible delay in service. As a batch exits the first stage, some may leave the system at that point with some probability. The rest of the batch attend an infinite buffer at the second stage with the complement of the probability of leaving. The attending batch es will be numbered and they will move to service as they are. As it can be anticipated for a time-dependent case, the system is a complicated one. Nonetheless, the time-dependent probability generating function for the number of jobs at each stage and the system as a whole, as well as the first and the second moments are found. The probability generation function and convolution of exponential functions and generating functions have been used to obtain moments for each stage as well as the system. Several special cases have been illustrated to show the validity of the results.

Keywords: Balking, batch, breakdown, bulk, delay, moments, probability generating function, Poisson, tandem time-dependent queue.

1. Literature Review and Introduction

Applications of queues with bulk arrival and batch service can be found in mass transportation such as buses and airplanes carrying passengers and trucks carrying all sorts of packages of groceries, cars, clothing, also groups of soldiers. Service in a queue by batches was presented by Bailey [3]. We, in this paper, view this type of queue from a

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management viewpoint. For instance, for a hiring process that requires multiple interviews in tandem. We, in this paper, choose only two stages.

Tandem queues, also, have many applications. Here are some examples: (i) in hospital emergency rooms, where patients are admitted by triage nurses before going on to have a number of medical tests and procedures, see Gans et al. [13], (ii) Haghighi and Mishev [16] studies a tandem queueing system with job-splitting, feedback, and blocking, and (iii) Haghighi and Mishev [20] include variety of applications in this book-chapter. There other authors who have studied tandem queues. For example, Le Gall [31] studied the stochastic behavior of networks of single server queues; Kim and Ayhan [28] studied tandem queues with subexponential service time distributions; Baruah et al. [4] considered two- stage queuing model where the server provides two stages of service one by one in succession; and so on. However, in most cases some authors study the stationary processes of the models they are presenting.

A general class of bulk queues with Poisson input was studied by Neuts [35]. Continuing that pattern, Kleinrock [29] considered models that are equivalent to Erlang arrival systems. That is, cases such as parallel and parallel-series services. Recently Abhishek et al. [1] studies the transient and stationary queue length distributions of a class of service systems with correlated service times. He stated in its abstract that the classical $M^X/G/1$ queue with semi-Markov service times is the most prominent example in this class. The idea of a queue with bulk arrival and batch service and breakdown in the stationary case was studied by Madan et al. [34]. Chen et al. [8] discuss Markovian bulk-arrival and bulk-service queues with general state-dependent control and answer the questions regarding hitting times and busy period distributions. A kind of generality of their model was studied by Jain et al. [25]. Using functional equation, Haghighi and Mishev [18] considered analysis of a two-node job-splitting feedback tandem queue with infinite buffers. Using functional equation, Haghighi et al. [19] considered a single-server Poisson queueing system with splitting and delayed-batch-feedback, the case that the batch size is 1. They considered this system as a tandem queue and offered an algorithm to find the solution.

The first author who considered the time-dependent case of a single server Markovian queue was Takács, who starded his paper in [40], completed and published it in [42]. Griffiths et al. [15] studied the transition solution of $M/E_k/1$ queue. Also, Hayashi et al. [23] studies queueing models, where customers arrive according to a continuous-time binomial process on a finite interval. A total of K customers arrives in a finite time interval. They introduce the auxiliary model with non-homogeneous Poisson arrivals and show that the time-dependent queue length distribution in the original model to analyze the time-dependent queue length distribution of this model.

And recently, Ayyappan and Udayageetha [2] consider a transient priority queuing model. Gahlawat et al. [12] consider a two-state time-dependent bulk queue model with intermittently available Server. They show that time between arrivals, servicing time, and server availability time follows an exponential distribution. Also, Stolletz [39] studied an optimization of time-dependent queueing system, in which the author decisions in service operations, manufacturing, and logistics are supported using stationary queueing models.

Study of a time-dependent tandem queue with blocking and a single server in each counter with no buffer between the stages was done by Prabhu [37]. In that case, the first station may be blocked if its server is busy at the end of a service at stage one. Hence, an idle period will be created and moving of a customer to the next station will be dependent upon the idle period. Thus, the process loses its Markovian property at such epochs. Thus, he concluded that although the idle period has a negative exponential distribution behavior and the state of station 1 will not change. Hence, he uses the first station as an $M/G/1$. Haghighi et al. [17] studied the transient probability distribution of a single-server queue with delay. They started by considering an $M/G/1$ with exponential and deterministic types of delayed service distributions. They found the busy period and the PGF of the queue length distribution. For the model they used, they considered the Erlang multi-stage distribution.

2. The Model

In this paper we study a time-dependent two-stage queueing network with variety of properties. Generally, there are two advantages of time-dependent analysis: (i) it will help us to understand the behavior of the system when the parameters involved are modified and (ii) it can contribute to the costs and benefits of operating the system.

In today's technologically oriented communication and manufacturing systems, performance evaluation is possible by modeling them as queueing networks. Parametric decomposition is one of the most popular and effective methods in queueing network approximations. It is a parametric decomposition which decomposes a network into single nodes and the interactions between these nodes are captured by a few parameters.

In addition to the time dependence, the model we are considering has the following features:

(1) arrivals to both stages are according to Poisson distribution, with bulks of various sizes,
(2) there is a possibility of balking at both stages,
(3) there is an infinite buffer in front of each stage,
(4) there is a single server in each stage,
(5) service is performed at each service station by a batch with limited size,
(6) there is a delayed process before service starts in S1,
(7) there is a possibility of service delay and breakdown in S1 that may require, and
(8) only a part of severed jobs exiting SS1 enter S2.

See Figure 1.

The goals of this study are finding the time-dependent probability generating function for the number of jobs in each stage and the in the entire system, as well as, the first and the second moments. We will also discuss the output of the first stage, and offer some special cases in both stages.

As a real-world application of the model consider a travel agency taking groups of tourists to different parts of the world for site seen and educational tourism. The process involves creating, organizing, and managing tours and experiences for tourists. Research and planning are the bedrock of any successful business. Thus, understanding the processing time and other aspect of the tour are important to determine the const of the travel for tourist and benefits for the company. Hence, what our model contribute in the calculation is to keep the management inform of the probability of how many tourist are in the tour at any time so that the company can calculate the expenses having the number to take care of, not only the traveling part about the wellness of the tourist know how long passing through traveling requirements such as checking passport, or going through gate , museum, etc. will take and, hence, the cost of hotel emergencies, etc. On the other hand, the income based on the number will help the management be able to estimate the profit at any time.

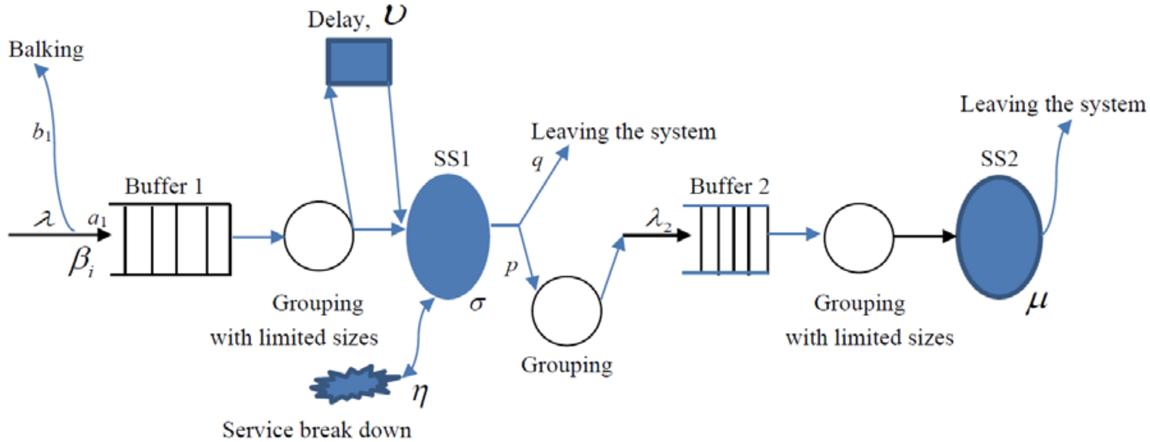

Figure 1. The Model.

Accordingly, our model shows the arrival of potential tourists via planes, ships, trains or busses to an airport, a port, a train station, or a bus station, of the city that the travel is to start, say Houston, Texas in the United States, with varying number of passengers, say, $i,\ i = 1, 2, 3, \cdots, n,\ \ n \to \infty$. Before a vessel arrives to its station, it may not be able to land or dock, or pulls into the station or arrive at the bus station due to natural disaster such as Southern California's wildfires (Palisades) in January 2025, terror activities, mechanical matter, etc. and thus will not be able to arrive and has to leave. Otherwise, arrives and walk into a specified terminal or are moved by shuttles, etc., after the plane has stopped. Then, they have to go to the gate to check their passports, etc., before going to the next pane to go for the destination. The travel process now follows the model described by Figure 1 and Figure 2.

Figure 2. Arrivals with possible balking.

Due to the limitations of closed form solutions and exact algorithms to analyze general queueing networks, approximations are the tools researchers are focusing on. In most such queueing systems, the approximations are used, usually, on a few parameters such as the interarrival and service time distributions and the first two moments. Hence, large errors in many conditions are possible. Girish and Hu [14] develop higher order approximations for the single server and tandem queues with infinite waiting buffer, general interarrival and service time distributions and focused on the phenomena of splitting, merging and feedback. Although manufacturing and communication systems have only finite buffers, they expressed their hopes that their results lead to solutions for the finite buffer systems in the future. They illustrated the gains and the accuracy of these methods and offered numerical examples.

For a network queue, and in particular, time-dependent ones, Haghighi and Mishev [21] have considered different types of *BMAP* queueing models in tandem. One of the problems in tandem queues is the arrival into the next station in series. For example, Beutler and Melamed [5] consider decomposition of customer stream of feedback networks queues in equilibrium.

In addition to the above characteristic of our model, there are now and in the recent past many studies of a class of queueing systems with catastrophes in which not only the servers fail, but all jobs are demolished and in some cases no arrival is allowed while servers are not back to serve. We will explain how our system works in S1(i) and S1(ii) below.

Thus, our model consists of two stages and the first stage has two cases. We will first discuss the first stage, referred to as the Stage 1 or S1, with both cases S1(i) and S1(ii), then the second stage, referred to as the Stage 2 or S2, and finally, the entire system.

3. Stage 1 (S1) Description

Jobs arrive according to Poisson distribution with mean rate λ, $\lambda > 0$ in batches of variable sizes that can be modeled as a point process. The batch sizes are represented by a random variable *X* with probability β_i, that is,

$$P\{X = i\} = \beta_i, \; i = 1, 2, 3, \cdots, \tag{3.1}$$

with mean value of batch sizes as $E(X) \equiv \bar{X}$. The balking occurs with probability b_1 and attendance occurs with probability a_1 such that $0 \le a_1,\ b_1 \le 1,\ a_1 + b_1 = 1.$ Thus, the batches attend according to a time-homogeneous compound Poisson process with parameter $\lambda a_1 \beta_i$, where, $0 \le \beta_i \le 1,$ and $\sum_{i=1}^{\infty} \beta_i = 1.$ We assume that both mean and variance of *X* are positive and finite.

An infinite size buffer, denoted by B1, is available for attending batches. Jobs exit an attending batch individually and are placed randomly in B1 with some types of identification for the purpose of servicing as first-come first-served.

Letting $\{\zeta_1(t),\ t \geq 0\}$ representing the number of jobs in S1 at time *t*, including the ones, if any, being served, results the process $\{\zeta_1(t),\ t \geq 0\}$ to be a Markov process on the state space.

S1 contains a service station, referred to as the Service Station 1 (or SS1) with a single server. The jobs in B1 are called, in the order of their placement (FCFS) and forming batches of varying sizes between a minimum *k* and a maximum *K*. That is, if there are *m* jobs available in B1, then the service works as follows:

(a) if $0 \leq m < k$, service will not start,

(b) if $k \leq m < K$, the server will pick up the entire *m* jobs and starts the servicing, and

(c) if $m \geq K$, the server will pick up *K* jobs and starts servicing.

It should be noted that the case $k = K = 1$ is a case of service being performed singly. If arrivals also arrive singly, then the S1 will be a time-dependent $M^{[X]}/M/1$ queue with balking. Some authors such as Liu and Song [33], who derived the probability generating function of the stationary queue length of such a model. However, we will present the time-dependent special case of $k = K = 1$ with a single-arrivals.

Service in S1 is performed independent of arrivals and with two types of possible delays as follows:

S1(i) As a batch enters SS1, before its service starts, it is required to go through **registration** and other clerical processes. The totality of these processes will cause a delay in service. For this case, we refer to Stage 1 as **S1(i)**. In this case, the external jobs continue to arrive while SS1 is going through the delay process without interruption. **The service time, however, will be the sum of the times of registration and the actual service**. Hence, the effect of interruption in this case is causing **delay** in the service time of jobs being in SS1.

S1(ii) The same as S1(i) except while a service is performing, either the server becomes disabled, or the machine experiences a **breakdown** and fails to continue to work. For this case, we refer to Stage 1 as **S1(ii)**. In this case, B1 closes and no job can arrive until the repair is done and the server is back to work. Thus, there is a possibility of the service **breakdown or server incapacitation** during the interval $(t, t+\Delta t]$ with probability $\omega \Delta t$. In this case, the effect of delay not only is on the sojourn time of the interrupted jobs in service and those in B1, but, also, on the number of jobs in S1, since some potential arrivals will be lost during the service breakdown. This type of phenomenon is almost unavoidable in real-life situations these days with all electronic equipment and Internet use. It is not practical that the server be available in the system on an enduring basis.

It should be noted that the model for the S1 that we have develped is similar to Madan [34] with two main differences. Firstly, he considered the stationary processes, while we are considering the time-dependent case. Secondly, he considered two separate probabilities for the numbers in queue; one when the server is offline and one when it is operational. However, he did not mention if his stem stops accepting arrivals while server is in repair, otherwise the distinction is unnecessary. But we do separate the cases as in S1(i) and S1(ii).

Stationary case of the S1 without either service delay or breakdown, which is an $M^{[X[}/M^{(k,K)}/1$, has been studied by Haghighi and Mishev [20]. Shanthi et al. [38] has considered the computational aspect of time-dependent $M^{[X]}/M^{(a,b)}/1$ with a standby server which is published in a proceeding of a conference. Lastly, Haghighi and Mishev [22] have considered a one-stage time-dependent case of $M^{[X]}/M^{(k,K)}/1$ with Reneging and Setup Time for Service.

4. Analysis of S1(i)

Let us denote by $P_m(t)$ the probability of the Stage 1 be in state m, $m \geq 0$, in case (i), where m is the number of jobs in B1 and in service, represented by a random variable $\varsigma_{1(i)}(t)$. Without loss of generality, we assume that initially there is no job in the system, that is,

$$P_j(0) = \delta_{0j} = \begin{cases} 1, & j = 0, \\ 0, & j \neq 0, \end{cases} \tag{4.1}$$

where δ_{0j} is the Kronecker delta.

Let us denote the service and the delay time by two independent random variables S and D, respectively. Then, the total time for a job to spend in SS1 is the sum of these two random variables that we denoted it by Y. That is,

$$Y = D + S. \tag{4.2}$$

We also assume that each of random variables S and D is exponentially distributed with parameters σ and υ, respectively, $\sigma > 0$, $\upsilon > 0$, and $y \geq 0$. Thus, the pdf of Y is the convolution of pdfs of S and D, which becomes the pdf of the **two-parameter hyperexponential distribution**. In other words, if we denote f_D and f_S as the pdf of D and S, respectively, defined as:

$$\begin{cases} f_D(t;\upsilon) = \upsilon e^{-\upsilon t}, & t \geq 0, \\ 0, & t < 0, \end{cases} \text{ and } \begin{cases} f_S(t;\sigma) = \upsilon e^{-\sigma t}, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{4.3}$$

then

$$\begin{aligned} f_{D+S}(t;\upsilon+\sigma) = f_Y(t;\upsilon+\sigma) &= \frac{\upsilon}{\upsilon-\sigma}\left(\sigma e^{-\sigma y}\right) + \frac{\sigma}{\sigma-\upsilon}\left(\upsilon e^{-\upsilon y}\right) \\ &= \frac{\upsilon\sigma}{\sigma-\upsilon}\left(e^{-\upsilon y} - e^{-\sigma y}\right). \end{aligned} \tag{4.4}$$

Also, since the expected value of a sum is the sum of the expected values, the expected value of a hyperexponential random variable with two parameters is:

$$E(Y) = \frac{1}{\upsilon} + \frac{1}{\sigma} = \frac{\upsilon+\sigma}{\upsilon\sigma}. \tag{4.5}$$

Note that if $\phi \equiv \upsilon = \sigma$, then Y becomes a two-parameter Erlang or gamma random variable, and from (4.5), with mean $2/\phi$. Hence, the parameter of pdf of Y in this special case will be $\phi/2$.

Now, we consider S1(i), that is, the time-dependent case of $M^X/M^{(k,K)}/1$ under conditions mentioned above. For the sake of convenience, we define the parameter

$$\psi \equiv \frac{\upsilon\sigma}{\sigma+\upsilon}. \tag{4.6}$$

Thus, the transient system of differential-difference equations (or the forward Kolmogorov equations) for the number of jobs governing S1(i) would be as follows:

$$P_0'(t) = -\lambda P_0(t) + \sum_{i=k}^{K} \psi P_i(t), \quad t \geq 0, \tag{4.7}$$

$$P_m'(t) = -\lambda P_m(t) + \sum_{i=1}^{m} \lambda a_1 \beta_i P_{m-i}(t) + \sum_{i=k}^{K} \psi P_{m+i}(t), \; 1 \leq m \leq k-1, \quad t \geq 0, \tag{4.8}$$

$$P_m'(t) = -\left(\lambda+\psi\right) P_m(t) + \sum_{i=1}^{m} \lambda a_1 \beta_i P_{m-i}(t) + \sum_{i=k}^{K} \psi P_{m+i}(t), \; m \geq k, \quad t \geq 0. \tag{4.9}$$

5. The Time-dependent Solution of the System of Equations for S1(i)

Theorem 5.1. *Let the probability generating function (PGF) of* $P_m(t)$, *represented by a random variable* $\varsigma_{1(i)}(t)$, *be denoted by* $G_{\zeta_{1(i)}}(t),\, t \geq 0$, *and define it as:*

$$G_{\zeta_{1(i)}}(w,t) = E\left(w^{\zeta_{1(i)}}\right) = \sum_{m=0}^{\infty} P\left\{\zeta_{1(i)}(t) = m\right\} w^m = \sum_{m=0}^{\infty} P_m(t) w^m, \qquad |w| \leq 1. \tag{5.1}$$

Then, under the condition that the S1(i) is empty at time 0, (4.1), the distribution of the number of jobs in S1(i) at any time t will be:

$$P_m(t) = \frac{1}{m!} G^{(m)}_{\xi_{1(i)}}(0,t) = \frac{1}{m!} \left. \frac{\partial^{(m)} G_{\xi_{1(i)}}(w,t)}{\partial w^m} \right|_{w=0}, \tag{5.2}$$

where $G_{\zeta_{1(i)}}(w,t)$ *is given as*:

$$G_{\zeta_{1(i)}}(w,t) = e^{-[\lambda+\psi-\lambda a_1 A(w)]t} \left\{ 1 + \psi \int_0^t e^{[\lambda+\psi-\lambda a_1 A(w)]u} \left[\sum_{m=0}^{k-1} P_m(u)\, w^m + \sum_{m=0}^{\infty} \sum_{i=k}^{K} P_{m+i}(u)\, w^m \right] du \right\}. \tag{5.3}$$

Proof. We first need to show that (5.3) is true. For this purpose, we will use the PGF to solve of the system (4.7) through (4.9). Thus, we multiplying (4.7) by w^K and multiplying (4.8) and (4.9) by w^m. Then, summing each product from 1 to ∞, with constraints on *m*. Then, with some calculations, we will have:

$$\frac{\partial G_{\xi_1(i)}(w,t)}{\partial t} = -\lambda G_{\xi_1(i)}(w,t) + \psi \sum_{m=k}^{K} P_m(t) - \psi \sum_{m=k}^{\infty} P_m(t)\, w^m + \lambda \sum_{m=1}^{\infty}\sum_{i=1}^{m} a_1 \beta_i P_{m-i}(t)\, w^m + \psi \sum_{m=1}^{\infty}\sum_{i=k}^{K} P_{m+i}(t)\, w^m$$

or

$$\frac{\partial G_{\xi_{1(i)}}(w,t)}{\partial t} + (\lambda+\psi) G_{\xi_1}(w,t) = \psi \sum_{m=k}^{K} P_m(t) + \psi \sum_{m=0}^{k-1} P_m(t)\, w^m + \lambda a_1 \sum_{m=1}^{\infty}\sum_{i=1}^{m} \beta_i P_{m-i}(t)\, w^m + \psi \sum_{m=1}^{\infty}\sum_{i=k}^{K} P_{m+i}(t)\, w^m. \tag{5.4}$$

We denote the PGF of β_i by $A(w)$, which is defined as:

$$A(w) = \sum_{i=1}^{\infty} \beta_i\, w^i = \beta_1\, w + \beta_2\, w^2 + \cdots, \tag{5.5}$$

from which

$$A(1) = \sum_{i=1}^{\infty} \beta_i = 1. \tag{5.6}$$

Hence, based on a well-known fact regarding product of two PGFs, we obtain:

$$\left[G_{\zeta_{1(i)}}(w,t)\right]\left[A(w)\right] = \sum_{m=0}^{\infty}\left[\sum_{i=0}^{m} \beta_i P_{m-i}(t)\right] w^m = \sum_{m=1}^{\infty}\sum_{i=1}^{m} \beta_i P_{m-i}(t) w^m = A(w) G^*_{\xi_1}(w,t). \tag{5.7}$$

Applying (5.7) on (5.4), and rearranging the terms leads to:

$$\frac{\partial G_{\zeta_{1(i)}}(w,t)}{\partial t} + \left(\lambda+\psi-\lambda a_1 A(w)\right) G_{\zeta_{1(i)}}(w,t) = \psi\left[\sum_{m=k}^{K} P_m(t) + \sum_{m=0}^{k-1} P_m(t) w^m + \sum_{m=0}^{\infty}\sum_{i=k}^{K} P_{m+i}(t) w^m\right]. \tag{5.8}$$

To solve (5.8), it should be noted that although it is a partial differential equation, it is really a linear ordinary differential equation with respect to t in the form of $y' + p(t)y = f(t)$, where:

$$p(t) = \lambda + \psi - \lambda A(w) \ \text{ and } \ f(t) = \psi\left[\sum_{m=k}^{K} P_m(t) + \sum_{m=0}^{k-1} P_m(t)\, w^m + \sum_{m=0}^{\infty}\sum_{i=k}^{K} P_{m+i}(t)\, w^m\right], \tag{5.9}$$

Thus, the integrating factor, denoted by ν, will be:

$$\nu = e^{\int_0^t \left(\lambda+\psi-\lambda a_1 A(w)\right) du} = e^{\left[\lambda+\psi-\lambda a_1 A(w)\right] t}. \tag{5.10}$$

Therefore, using (5.10), the solution of (5.8) will be:

$$G_{\zeta_{1(i)}}(w,t)=e^{-[\lambda+\psi-\lambda a_1 A(w)]t}\left\{c+\psi\int_0^t e^{[\lambda+\psi-\lambda a_1 A(w)]u}\left[\sum_{m=0}^{k-1}P_m(u)\,w^m+\sum_{m=0}^{\infty}\sum_{i=k}^{K}P_{m+i}(u)\,w^m\right]du\right\}, \quad (5.11)$$

where c in an arbitrary constant.

Using the fact that $G_{\xi_{1(i)}}(1,t)=\sum_{m=0}^{\infty}P_m(t)=1,$ and the initial condition at $t=0,$ that is, (4.1), equation (5.11) leads to

$$G_{\xi_{1(i)},}(1,0)=\sum_{m=0}^{\infty}P_m(0)=1. \quad (5.12)$$

Applying (5.12) on (5.11), we will have:

$$G_{\zeta_{1(i)}}(1,0)=1=c. \quad (5.13)$$

Substituting c from (5.13) into (5.11), we obtain (5.3), from which (5.2) will be obtained and the proof is completed.

6. Moments of Number of Jobs in S1(i)

Now that we have the PGF of the time-dependent S1(i) given by Theorem 5.1, we move to find the first two moments of $\zeta_{1(i)}(t)$. The mean and variance of the number of jobs in S1(i) can be found by the first and the second derivatives of (5.3) with respect to w, and being evaluated at $w=1,$ that is,

$$E\left(\xi_{1(i)}(t)\right)=G'_{\xi_{1(i)}}(1,t)=\left.\frac{\partial G_{\xi_{1(i)}}(w,t)}{\partial w}\right|_{w=1} \text{ and } E\left(\xi^2_{1(i)}(w,t)\right)=\left.\frac{\partial^2 G_{\zeta_{1(i)}}(w,t)}{\partial w^2}\right|_{w=1}. \quad (6.1)$$

In other words,

$$E\left(\zeta_{1(i)}(t)\right)=G_w(1,t) \text{ and } Var\left(\zeta_{1(i)}(t)\right)=G_{ww}(1,t)+G_w(1,t)-\left(G_w(1,t)\right)^2 \quad (6.2)$$

Theorem 6.1. *The mean of the number of jobs in* S1(i) *in case of the time dependent case is:*

$$E\left(\zeta_{1(i)}\right)=e^{-\psi t}\left\{\lambda a_1\bar{X}\,t\left[1+\int_0^t e^{\psi u}R_1(1)du\right]-\int_0^t e^{\psi u}\left[\lambda a_1\bar{X}\psi R_1(1)-R_1'(1)\right]du\right\}, \quad (6.3)$$

where

$$R_1(1)=\psi\left[P_0(u)+\sum_{m=1}^{\infty}\sum_{i=k}^{K}P_{m+i}(u)+\sum_{m=1}^{k-1}P_m(u)+\sum_{i=k}^{K}P_i(u)\right], \quad (6.4)$$

and

$$R_1^{'}(1)=\psi\left[\sum_{m=1}^{\infty}\sum_{i=k}^{K}mP_{m+i}(u)+\sum_{m=1}^{k-1}mP_m(u)\right]. \qquad (6.5)$$

Also, the second moment of the number of jobs in S1(i) in case of the time dependent case is:

$$\left.\frac{\partial^2 G_{\zeta_{1(i)}}}{\partial w^2}\right|_{w=1}=e^{-\psi t}\left\{\left[\left(\lambda a_1\bar{X}\,t\right)^2-\sum_{i=1}^{\infty}i(i-1)\alpha_i\right]\cdot\left[1+\int_0^t e^{\psi u}R_1(1)du\right]\right.$$
$$-2\lambda a_1\bar{X}\,t\int_0^t e^{\psi u}\left[\lambda a_1\bar{X}uR_1(1)-R_1^{'}(1)\right]du$$
$$\left.+\int_0^t e^{\psi u}\left[\left(\lambda a_1\bar{X}\,u\right)^2R_1(1)+\left(\sum_{i=1}^{\infty}i(i-1)\alpha_i\right)uR_1(1)-2\left[\lambda a_1\bar{X}uR_1^{'}(1)-R_1^{''}(1)\right]\right]du\right\}, \qquad (6.6)$$

where

$$R_1^{''}(1)=\psi\left[\sum_{m=1}^{\infty}\sum_{i=k}^{K}m(m-1)P_{m+i}(u)+\sum_{m=1}^{k-1}m(m-1)P_m(u)\right]. \qquad (6.7)$$

The variance can be obtained using (6.2).

Proof. To find the derivatives, we need some manipulations such as:

$$-\left[\lambda+\psi-\lambda a_1A(w)\right]t+[\lambda(1-a_1)+\psi]t=\left(-\lambda-\psi+\lambda a_1A(w)+\lambda(1-a_1)+\psi\right)t$$
$$=\left(\lambda a_1A(w)-\lambda a_1\right)t=-\lambda a_1(1-A(w))t,$$

which leads to:

$$e^{-\left[\lambda+\psi-\lambda a_1A(w)\right]t}e^{[\lambda(1-a_1)+\psi]t}=e^{-\lambda a_1(1-A(w))t}.$$

Since:

$$A(1)=\sum_{i=1}^{\infty}\beta_i=1 \quad \text{and} \quad A'(1)=\sum_{i=1}^{\infty}i\beta_i=\bar{X},$$

we will have,

$$\left.\frac{\partial e^{-\left[\lambda+\psi-\lambda a_1A(w)\right]t}}{\partial w}\right|_{w=1}=-\left[\frac{\partial\left[\lambda+\psi-\lambda a_1A(w)\right]t}{\partial w}\right]e^{-\left[\lambda+\psi-\lambda a_1A(w)\right]t}=\frac{dA(w)}{dw}\left.\left(\lambda a_1te^{-\left[\lambda+\psi-\lambda a_1A(w)\right]t}\right)\right|_{w=1}$$
$$=\lambda a_1\bar{X}t\,e^{-\left[\lambda+\psi-\lambda a_1\right]t}.$$

Hence,

$$E\left(\xi_{1(i)}(t)\right) = G'_{\xi_1(i)}(1,t) = \left.\frac{\partial G_{\xi_{1(i)}}(w,t)}{\partial w}\right|_{w=1}$$

$$= \frac{\partial}{\partial w}\left[e^{-[\lambda+\psi-\lambda a_1 A(w)]t} \left\{ 1 + \psi \int_0^t e^{[\lambda+\psi-\lambda a_1 A(w)]u} \left[\sum_{m=0}^{k-1} P_m(u)\, w^m + \sum_{m=0}^{\infty}\sum_{i=k}^{K} P_{m+i}(u)\, w^m \right] du \right\} \right]_{w=1}$$

$$= \left.\frac{\partial}{\partial w} e^{-[\lambda+\psi-\lambda a_1 A(w)]t}\right|_{w=1}$$

$$+ \frac{\partial}{\partial w}\left[\psi e^{-[\lambda+\psi-\lambda a_1 A(w)]t} \int_0^t e^{[\lambda+\psi-\lambda a_1 A(w)]u} \left[\sum_{m=0}^{k-1} P_m(u)\, w^m + \sum_{m=0}^{\infty}\sum_{i=k}^{K} P_{m+i}(u)\, w^m \right] du \right]_{w=1},$$

or

$$E\left(\xi_{1(i)}(t)\right) = \left.\frac{\partial G_{\zeta_{1(\mathrm{i})}}(w,t)}{\partial w}\right|_{w=1} = \lambda a_1 \bar{X} t\, e^{-[\lambda+\psi-\lambda a_1]t}$$

$$+ \frac{\partial}{\partial w}\left[\psi e^{-[\lambda+\psi-\lambda a_1 A(w)]t} \int_0^t e^{[\lambda+\psi-\lambda a_1 A(w)]u} \left[\sum_{m=0}^{k-1} P_m(u)\, w^m + \sum_{m=0}^{\infty}\sum_{i=k}^{K} P_{m+i}(u)\, w^m \right] du \right]_{w=1}.$$

Thus, after some more manipulations, algebra and taking the second derivative, the proof will be completed.

7. The Steady-state Solution of the System of Equations for S1(i)

In Section 5, thorough Theorem 5.1, we found the transient PGF of the number of items in S1 for the case S1(i). Now we want to let *t* to approach infinity and find the steady-state PGF of the same. This is the case that the derivatives on the left-hand side of the system (4.7) – (4.9) are to be zero and *t* should be dropped on the right-hand side of it.

Thus, for such a case, we can find P_0 from the first equation after the derivative on the let is set to zero, that is:

$$P_0 = \frac{\psi}{\lambda} \sum_{i=k}^{K} P_i. \tag{7.1}$$

The stationary case of S1(i) is similar to the first station of Haghighi and Mishev [20], Stochastic three-stage hiring, $M^{[X]}/M^{(k,K)}/1-M^{[Y]}/E_r/1-\infty$, equation (3.8) Going through the derivation of (3.8) for our case with balking, we will have the stationary PGF for our case S1(i) will be as follows:

$$G_{\zeta_{1(\mathrm{i})}}(w) = \frac{\psi\left\{ w^K \sum_{m=0}^{k-1} P_m w^m - \left(\sum_{j=k}^{K} w^{K-j} \sum_{i=0}^{j-1} P_i\, w^i \right) \right\}}{w^K\left(\lambda + \psi - \lambda a_1 A(w)\right) - \psi\left(\sum_{i=k}^{K} w^{K-i} \right)}. \tag{7.2}$$

where

$$G_{\zeta_{1(i)}}(w) = \sum_{m=0}^{\infty} P_m w^m, \qquad |w| \le 1, \tag{7.3}$$

Finite sums in the numerator of (7.2) may be found using the standard argument of applying Rouche's theorem. That is, knowing $A(w)$, the number of zeros of denominator of (7.2) within and on the unit- circle is the same as the ones of the numerator. The application of Rouche's theorem assures us that the denominator of (7.2), keeping in mind that $\sum_k^K 1 = K - k + 1,$ has $K - k + 1$ roots on or inside the unit circle $|w| = 1.$ Thus, the numerator must have the same roots. Hence, we will have K linear equations in terms of $w_i,\ i = 0,1,2,\cdots,K,$ which are sufficient to determine all the K unknowns. Therefore, the probability generating functions in (7.2) can be completely determined, from which mean and variance in this case can be obtained. See, for example, Madan [34].

8. Special Cases of S1(i)

8.1. Case when $k = K = 1$ with single arrivals

For a special case $k = K = 1,$ we consider two cases: (a) no balking and (b) with balking.

(a) Case when $k = K = 1$, with single arrivals and no balking

The difference between our case and the standard transient *M*/*M*/1 is the service rate due to the delay factor. Hence, using the known information about *M*/*M*/1, we can find the distribution of the queue size for this special case of S1(i). Information about the standard *M*/*M*/1 system can be found in Takács [41, 43], which contains complete discussion of transient *M*/*M*/1, including busy period, waiting time, and the queue size. Also, Jain and Meitei [25] discussed computation of the transient solution of *M*/*M*/1 queue, as was mentioned earlier. We will use their method in this part.

Now, let $M(t)$ denote the number of jobs in S1(i) at time *t*. Assume that the initial numbers of jobs in S1 at $t = 0$ is *i*, that is, in B1 and SS1. In other words,

$$P_i(0) = \delta_{0i} = \begin{cases} 0, & i = 0, \\ 1, & i \ne 0, \end{cases} \tag{8.1}$$

where δ_{0i} is the Kronecker delta. Then, using the transition probabilities $P_{ij}(t) = P\{M(t) = j \mid M(0) = i\}$, the probability generating function, the Laplace transform, the renewal structure, and the busy period density, Kijima [27] shows that there is a probability density function, say $p(t)$, such that:

$$P_{00}(t) = 1 - \rho \int_0^t p(u)du\,, \tag{8.2}$$

where:

$$\rho = \frac{\lambda \bar{X}}{\psi} < 1. \tag{8.3}$$

Also, for the case $k = K = 1$, S1(i) becomes a time-dependent *M*/*M*/1, whose solution is well known. Takács was the first to consider this case in his book [43], where he offers the transition distribution function for the number of jobs is the system at time *t* with initial number *i*. These findings can be found in his Theorem 1 in term of integral and in Theorem 2 in term of the modified Bessel function of order *r*, which is given as:

$$I_r(z) = \sum_{l=0}^{\infty} \frac{1}{(l+1)!(l+r)!} \left(\frac{z}{2}\right)^{2l+r}, \quad r = 0,1,2,\cdots, \tag{8.4}$$

where *l* and *z* are a real and a complex number, respectively, in Chapter 1, pages 22 - 26. Further, Leguesdron et al. [32] study the same case but with a different approach through PGF and its inversion. And Brockwell [6] considered a similar case when the batch sizes are fixed, and they are served as they arrive. With that assumption, he found the probability of the number in the system using the output distribution and the distribution of the system being empty, its Lagrange expansion and the method of continued fractions. Jain and Meitei [25] consider the computation of the transient solution of *M*/*M*/1. Based on their solution, for our special case, the solution is as follows:

$$\begin{aligned} P_{im}(t) = e^{-(\lambda+\psi)t} \rho^{\frac{m-j}{2}} \left[I_{m-i}\left(2\sqrt{\lambda\psi t}\right) - I_{m+i}\left(2\sqrt{\lambda\psi t}\right) \right] \\ + e^{-(\lambda+\psi)t} \rho^{m} \sum_{l=m+i+1}^{\infty} \frac{l \rho^{\frac{-l}{2}}}{\psi t} I_l\left(2\sqrt{\lambda\psi t}\right), \end{aligned} \tag{8.5}$$

where ρ is given in (8.3) and $I_m(z)$ is the modified Bessel function of order *m*, given in (8.4). After applying some properties of the modified Bessel function,

(b) Case when $k = K = 1$, with single arrivals and with balking

Parthasarathy [36], based on Ledermann and Reuter [30], the combinatorial method of Champernowne [10] and the difference equation technique of Conolly [11], in a two-page paper, presented the distribution of the number in the system for standard *M*/*M*/1 in a simple way. Based on his method, and our assumptions of *M*/*M*/1 with delayed service rate $\psi \equiv \upsilon\sigma / (\sigma + \upsilon)$, and balking, the system of difference equations (4.7) to (4.9) reduces to the following system:

$$P_0^{'}(t) = -\lambda P_0(t) + \psi P_1(t), \quad t \geq 0, \tag{8.6}$$

$$P_m^{'}(t) = -\left(\lambda + \psi\right) P_m(t) + \lambda a_1 P_{m-1}(t) + \psi P_{m+1}(t), \quad t \geq 0, \quad m = 1,2,\cdots, \tag{8.7}$$

For this case, we assume that initially there are *i* jobs in S1, as was defined in (8.1). Thus, after going through the PGF manipulation with his way, we find the following result for our case:

$$P_m(t) = \frac{1}{\psi} e^{-(\lambda+\psi)t} \sum_{l=1}^{m} q_l(t) \left(\frac{\lambda}{\psi}\right)^{m-l} + \left(\frac{\lambda}{\psi}\right)^{m} P_0(t), \tag{8.8}$$

where

$$P_0(t) = \delta_{0i} + \int_0^t q_1(u) e^{-(\lambda+\psi)u} du, \tag{8.9}$$

δ_{0i} Kronecker delta defined in (8.1), and:

$$q_m(t) = \begin{cases} e^{(\lambda+\psi)t}\left[\psi P_m(t) - \lambda a_1 P_{m-1}(t)\right], & m = 1,2,3,\cdots, \\ 0, & m = 0, \end{cases}$$

$$\begin{aligned} &= \psi\left(\sqrt{\frac{\lambda}{\psi}}\right)^{m-i} (1-\delta_{0i})\left[I_{m-i}\left(2\sqrt{\lambda\psi}\, t\right) - I_{m+i}\left(\sqrt{\frac{\lambda}{\psi}}\, t\right)\right] \\ &\qquad + \lambda\left(\sqrt{\frac{\lambda}{\psi}}\right)^{m-i-1}\left[I_{m+i+1}\left(2\sqrt{\lambda\psi}\, t\right) - I_{m-i-1}\left(2\sqrt{\lambda a_1 \psi}\, t\right)\right], \end{aligned} \tag{8.10}$$

where $I_r(x)$ is the modified Bessel function of the first kind with $I_{-r} = I_r,\ n = 1,2,\cdots$.

9. Analysis of S1(ii)

In case of S1(ii), as mentioned above, when a breakdown occurs or the server becomes incapacitated, the service of jobs in SS1 stops and resumes as soon as it is repaired, or the server's condition is back to normal. No job can enter SS1 during the breakdown. Hence, we assumed that the **delay times** in this case are *iid* random variables according to an **exponential** distribution with parameter $\eta > 0$.

Let $\left\{\zeta_{1(ii)}(t),\ t \geq 0\right\}$ be the random variable represent the number of jobs in S1(ii) at time *t*. Let us, also, define $O_m(t)$ and $B_m(t)$ as follows:

1. Let $O_m(t)$ be defined as the probability of S1(ii) in state *m*, $m \geq 0$, where *m* is the number of jobs in B1 and the batch being served, when the service process is operational. Without loss of generality, we assume that initially there is no job in the system, that is,

$$O_m(0) = \begin{cases} 1, & m = 0, \\ 0, & m \geq 1. \end{cases} \tag{9.1}$$

2. Let us also define $B_m(t)$ as the probability of S1(ii) be in state $m,\ m \geq 0,$ where *m* is the number of jobs in B1 and the batch being processed, when the service process experiencing a breakdown. We also assume that:

$$B_m(0) = \begin{cases} 1, & m = 0, \\ 0, & m \geq 1. \end{cases} \tag{9.2}$$

The governing system of the differential-difference equations with respect to time *t*, in this case for S1(ii) would be as follows:

$$O_0^{'}(t) = -(\lambda + \omega)O_0(t) + \sum_{i=k}^{K} \sigma O_i(t) + \eta B_0(t),\ t \geq 0, \tag{9.3}$$

$$O_m^{'}(t) = -(\lambda + \omega)O_m(t) + \sum_{i=1}^{m} \lambda \beta_i O_{m-i}(t) + \sum_{i=k}^{K} \sigma O_{m+i}(t) + \eta B_m(t),\ 1 \leq m \leq k-1,\ \ t \geq 0, \tag{9.4}$$

$$O_m^{'}(t) = -(\lambda + \sigma + \omega)O_m(t) + \sum_{j=1}^{m} \lambda \beta_i O_{m-i}(t) + \sum_{i=k}^{K} \sigma O_{m+i}(t) + \eta B_m(t),\ m \geq k,\ t \geq 0, \tag{9.5}$$

$$B_0^{'}(t) = -(\lambda + \eta)B_0(t) + \omega O_0(t),\ t \geq 0, \tag{9.6}$$

$$B_m^{'}(t) = -(\lambda + \eta)B_m(t) + \omega O_m(t) + \sum_{i=1}^{m} \lambda \beta_i B_{m-i}(t),\ m \geq 1,\ t \geq 0. \tag{9.7}$$

To interpret the system (9.3) through (9.8), for instance, (9.3) means that for the system in working condition to have a 0 job in it, no job should arrive, no breakdown should occur, if there are *i* jobs, $i = k, k+1, \cdots, K,$ all have to be served and out of Station 1, and no repair should be in process. The rest may be interpreted the same way.

Note that, in a special case of time-dependent when arrival is singly according to Poisson with parameter λ, service is one at a time, that is, $k = K = 1$ and there is no breakdown possible, then $\beta_i = 1, i = 1, 2, \cdots,$ $\lambda_1 = \lambda,$ $\eta = 0,$ and $B_m(t) = 0.$ Hence, the time-dependent system of equations (9.3) – (9.7) becomes the system of equation for time-dependent *M*/*M*/1, solution of which was first given by Takács [40, 42].

To solve the system of time-dependent equations (9.3) through (9.7), we multiply (9.3), (9.4), and (9.6) by w^m and sum over *m* from 1 to ∞. Then, apply the Laplace transform on the new equations, using the following fact:

$$O_0(0) = 1 \text{ implies that } O_m(0) = 0,\ \ B_0(0) = 1,\ \text{and } B_m(0) = 0, \tag{9.8}$$

we obtain the following:

$$(s + \lambda + \omega)O_0^*(s) - 1 = \sigma \sum_{i=k}^{K} O_i^*(s) + \eta B_0^*(s), \tag{9.9}$$

$$(s+\lambda+\omega)\sum_{m=1}^{k-1} O_m^*(s)\, w^m = \lambda\sum_{m=1}^{k-1}\sum_{j=1}^{m} \beta_j O_{m-j}^*(s)\, w^m + \sigma\sum_{m=1}^{k-1}\sum_{j=k}^{K} O_{m+j}^*(s)\, w^m + \eta\sum_{m=1}^{k-1} B_m^*(s)\, w^m, \quad (9.10)$$

$$\begin{aligned}(s+\lambda+\sigma+\omega)\sum_{m=k}^{\infty} O_m^*(s)\, w^m &= \lambda\sum_{m=k}^{\infty}\sum_{j=1}^{m} \beta_j O_{m-j}^*(s)\, w^m \\ &\quad + \sigma\sum_{m=k}^{\infty}\sum_{j=k}^{K} O_{m+j}^*(s)\, w^m + \eta\sum_{m=k}^{\infty} B_m^*(s)\, w^m,\end{aligned} \quad (9.11)$$

$$(s+\lambda+\eta)B_0^*(s) - 1 = \omega O_0^*(s), \quad (9.12)$$

$$(s+\lambda+\eta)\sum_{m=1}^{\infty} B_m^*(s)\, w^m = \omega\sum_{m=1}^{\infty} O_m^*(s)\, w^m + \lambda\sum_{m=1}^{\infty}\sum_{j=1}^{m} \beta_j B_{m-j}^*(s)\, w^m. \quad (9.13)$$

To solve the system (9.9) through (9.13), as standard, we define the probability generating functions of $O_m^*(s)$, and $B_m^*(s)$, respectively, as follows:

$$G_{\zeta_{1(\mathrm{ii})O}}^*(w,s) = \sum_{m=0}^{\infty} O_m^*(s) w^m, \quad \text{and} \quad G_{\zeta_{1(\mathrm{ii})B}}^*(w,s) = \sum_{m=0}^{\infty} B_m^*(s) w^m, \quad |w|<1, \quad (9.14)$$

with $A(w)$ and $A(1)$ defined in (5.5) and (5.6), and

$$G_{\xi_{1(\mathrm{ii})O}}^*(w,1) = 1, \text{ and } G_{\zeta_{1(\mathrm{ii})B}}^*(w,1) = 1. \quad (9.15)$$

Thus, using the aforementioned on the system (9.9) through (9.13), we will have:

$$\begin{aligned}G_{\zeta_{1(\mathrm{ii})O}}^*(w,s) &= \frac{(\lambda+s+\eta)\lambda A(w)}{\big((\lambda+s+\eta)\lambda A(w)\big)(s+\sigma-\lambda) - \omega\big[\lambda(A(w)+1)+s\big]} \\ &\quad \times\left(-\frac{\big[\lambda(A(w)+1)+s\big]}{\lambda[A(w)+1]+s+\eta} + \sigma\sum_{m=0}^{k-1} O_m^*(s)\, w^m\right) \\ &\quad + \lambda\sum_{m=1}^{k-1}\sum_{j=1}^{m} \beta_j O_{m-j}^*(s)\, w^m + \eta\frac{1+\omega O_0^*(s)}{(s+\lambda+\eta)} + 2,\end{aligned} \quad (9.16)$$

and

$$G^*_{\zeta_{1(\text{ii})B}}(w,s) = \frac{1}{s+\lambda+\eta+\lambda A(w)} + \left(\frac{\omega}{s+\lambda+\eta+\lambda A(w)}\right) \times \left[\begin{array}{r} \dfrac{(\lambda+s+\eta)\lambda A(w)}{\big((\lambda+s+\eta)\lambda A(w)\big)(s+\sigma-\lambda)-\omega\big[\lambda(A(w)+1)+s\big]} \\ \times\left(-\dfrac{\big[\lambda(A(w)+1)+s\big]}{\lambda[A(w)+1]+s+\eta}+\sigma\sum_{m=0}^{k-1}O^*_m(s)\,w^m\right) \\ +\lambda\sum_{m=1}^{k-1}\sum_{j=1}^{m}\beta_j O^*_{m-j}(s)\,w^m+\upsilon\dfrac{1+\omega O^*_0(s)}{(s+\lambda+\eta)}+2 \end{array} \right]. \tag{9.17}$$

Again, the Laplace transform of mean and variance of the number of jobs in S1(ii) can be found by the first and the second derivatives of (9.16) and (9.17) with respect to *w*, evaluated at $w=1$, according to their relations.

As a final assumption on Stage 1, we assume that the inter-arrival times of batches, the service times of batches, the delay times and the repair times are all independent of each other. With the new assumption, the first stage becomes an $M^X/M^{(k,K)}/1$ with delayed service time due to the possible breakdown.

10. The Output Process of S1

The case of a tandem queues, generally, the arrivals into the next stage are the output of the immediate previous stage. In our model, arrivals could be from either S1(i) or S1(ii). There are cases when the new external arrival for each stage is possible. Thus, generally, the distribution of arrivals to the next stage is not known, unless, otherwise, it is either proven or assumed. For example, for a stationary tandem *M*/*M*/1, Burke [7] has proved that the output of the first stage is Poisson with the same rate as the arrival. Also, depending upon the service distribution being exponential or unknown, the model could be a *G*/*M*/*x*, *GI*/*M*/*x*, or *G*/*G*/*x*, where *x* is the number of servers. There is a variety of these types of models. For example, Takács[43], Kleinrock [29], Chen and Whitt [9] and Hora [24].

For our case, however, there might be an attempt to accept the output distribution from S1 as Poisson, based on Burke's [7] theorem and Takács [43] that essentially is for stationary *M*/*M*/1 and *M*/*M*/*c* from infinite sources. However, recently, Tsitsiashvili and Osipova [45] defined a time-dependent model $A_n \equiv M/M/n/\infty$ as a queueing system with input flow intensity $\lambda > 0$, and the service time with parameter $\mu > 0$, where $1 \le n < \infty$. They defined $P_{k,n}(t)$, $k \ge 0$, to be the transition probability from state k to state n at the epoch *t*. They, then, proved that the output flow from A_n is Poisson with intensity $a(t) = \sum_{k>0} \mu P_{k,n}(t)\min(k,n)$. Thus, they have given a generalization of the Burke's theorem. In other words, the output distribution is yet Poisson, but with different parameters.

Although, it seemed as though the model $M/M/n/\infty$ of Tsitsiashvili and Osipova is to help our model to assume the arrival to S2 as a compound Poisson distribution, but in fact, that is not the case. What Tsitsiashvili and Osipova are doing is making the Poisson arrival process a non-homogenous rather than the stationary case of Burke or Takács. Thus, we will try a different way.

11. The Second Stage (Stage 2 or S2)

The second stage starts as the jobs exit SS1. We referred to this stage as **Stage 2** or **S2**. There is an infinite-sized **buffer** at the beginning of S2, that we refer to as **B2**. Now, after service of a batch in SS1 is completed and the batch is to exit, some jobs may have to leave the system for whatever reason, with probability q_2. Hence, the output from S1 constitutes **potential arrivals** for S2. At this point, the remaining jobs will attend B1, with probability p_2, $q_2 + p_2 = 1$, as a batch of a varying size, represented by the variable *Q* with probability randomly with probability π_j, $j = 1, 2, \cdots, K$, that is,

$$P\{Q = j\} = \pi_j, \quad j = 1, 2, 3, \cdots, K, \tag{11.1}$$

with mean value of batch sizes as $E(Q) \equiv \bar{Q}$. When a batch attends B1, it will be placed with an identification tag for the service since the service will be first-come first-served.

There is also a service station in S2, referred to as **SS2**. Jobs batches in B2 will be called in order of their arrivals, and move to SS2 for service. The **service times in** SS2 are *iid* random variables having negative **exponential** distribution, $H(\theta)$, with parameters μ where μ is a positive real number, that is,

$$H(\theta) = \begin{cases} 1 - e^{-\mu\theta}, & \theta \geq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{11.2}$$

and independent of $\{\tau_n\}$. Thus, **Stage 2 is an infinite-buffer time-dependent $G^{[j]} / M^{[j]} / 1, \quad j = 1, 2, 3, L, K,$ type queueing model**. This model is a generalization form of *GI*/*M*/1 that many authors have addressed. For example, see Tian and Zhang [44]. Also, Zhao [46] has considered the steady-state case of $GI^X / M / c$.

12. Analysis of Stage 2

Similar to S1, for S2, let $\zeta_2(t)$, $t \geq 0$, be the random variable representing the number of jobs in S2 (the queue size) at time epoch *t*, including the batch being served; that is, in buffers and in the service station. Hence, the process $\{\zeta_2(t), t \geq 0\}$ forms a non-Markov process on the state space.

Since "*G*" in $G^{[j]} / M^{[j]} / 1$stands for an arbitrary general distribution, we choose it as a Poisson. The authors have presented their justification of this choice in Haghighi and Mishev [22]. We assume the arrival rate λ_2 Also, [*j*] in this case stands for a batch of fixed size *j*. Thus, now the system becomes $M^{[j]} / M^{[j]} / 1$, not quite similar to the case of S1. Hence, let us denote by $P_n(t)$ the probability of the number of jobs, *n*, in Stage 2, that is S2 be in state n, $n \geq 0$, where n is the number of jobs in B2 and in service, represented by a random variable $\varsigma_2(t)$. For this case, we assume that at time 0, there is l jobs in S2, that is,

$$P_l(0) = \delta_{0l} = \begin{cases} 0, & l = 0, \\ 1, & l \neq 0, \end{cases} \tag{12.1}$$

where δ_{0l} is the Kronecker delta. The process $\{\zeta_2(t), t \geq 0\}$ is a homogenous Markov chain. Thus, the system of differential difference equations under these assumptions mentioned will be:

$$P_0^{'}(t) = -\lambda_2 P_0(t) + \sum_{j=1}^{K} \mu P_j(t), \ \ t \geq 0, \tag{12.2}$$

$$P_n^{'}(t) = -\left(\lambda_2 + \mu\right) P_n(t) + \sum_{j=1}^{n} \lambda_2 \pi_j P_{n-j}(t) + \sum_{j=1}^{K} \mu P_{n+j}(t), \quad n \geq 1, \ \ t \geq 0. \tag{12.3}$$

Theorem 12.1. *Let the PGF of* $P_n(t) \equiv P\{\zeta_2(t) = n\}$, *denoted by* $G_{\varsigma_2(t)}(v,t)$, *be defined as*:

$$G_{\varsigma_2(t)}(v,t) = \sum_{n=0}^{\infty} P_n(t) v^n = P_0(t) + P_1(t)v + P_2(t)v^2 + \cdots + P_n(t)v^n + \cdots, \tag{12.4}$$

The PGF of the probability of the number of jobs in S2, *represented by the random variable* $\zeta_2(t), t \geq 0$, *that is,* $P_n(t)$, $P_n(t) \equiv P\{\zeta_2(t) = n\}$, *is obtained from the governing system of differential difference equations for* S2 *given by* (12.2) *and* (12.3) *as follows*:

$$G_{\varsigma_2(t)}(v,t) = e^{-\left[\lambda_2(1-F(v)) + \mu\left(1-\sum_{j=1}^{K}\frac{1}{v^j}\right)\right]} \Bigg\{ 1 + \int_0^t e^{\left[\lambda_2(1-F(v)) + \mu\left(1-\sum_{j=1}^{K}\frac{1}{v^j}\right)\right]u} \Bigg[\mu\left(1 - \sum_{j=1}^{K} \frac{1}{v^j}\right) \\ - \mu \sum_{j=1}^{K} \left[\frac{P_0(u)}{z^j} + \frac{P_1(u)}{z^{j-1}} + \frac{P_2(u)}{z^{j-2}} + \cdots + \frac{P_{j-1}(u)}{z} + P_j(u) \right] \Bigg] du \Bigg\}, \tag{12.5}$$

where

$$F(v) = \sum_{j=1}^{\infty} \pi_j v^j, \ \ |z| < 1. \tag{12.6}$$

Proof. We solve the system (12.2) and (12.3) using the PGF defined in (12.4), whose first partial derivative with respect to *t* is:

$$\frac{\partial}{\partial t}G_{\varsigma_2(t)}(v,t)=\sum_{n=0}^{\infty}P_n^{'}(t)v^n=P_0^{'}(t)+P_1^{'}(t)v+P_2^{'}(t)v^2+\cdots+P_n^{'}(t)v^n+\cdots. \tag{12.7}$$

Now, we multiply both sides of (12.3) by v^n and sum from 1 to n, as appropriate to obtain:

$$\sum_{n=1}^{\infty}P_n^{'}(t)v^n=-(\lambda_2+\mu)\sum_{n=1}^{\infty}P_n(t)v^n+\lambda_2\sum_{n=1}^{\infty}\sum_{j=1}^{n}\pi_jP_{n-j}(t)v^n+\mu\sum_{n=1}^{\infty}\sum_{j=1}^{K}P_{n+j}(t)v^n. \tag{12.8}$$

Similar to (5.7), we write:
and

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n}\pi_jP_{n-j}(t)v^n=F(v)G_{\varsigma_2(t)}(v,t). \tag{12.9}$$

Thus, simplifying and rewriting (12.8), we will have:

$$\begin{aligned}&\frac{\partial G_{\varsigma_2(t)}(v,t)}{\partial t}+\lambda_2P_0(t)-\sum_{j=1}^{K}\mu P_j(t)\\&=-(\lambda_2+\mu)\left[G_{\varsigma_2(t)}(v,t)-P_0(t)\right]+\lambda_2F(v)G_{\varsigma_2(t)}(v,t)\\&\qquad+\sum_{j=1}^{K}\frac{\mu}{v^j}\left[G_{\varsigma_2(t)}(v,t)-P_0(t)-P_1(t)v+\cdots+P_j(t)v^j\right],\end{aligned}$$

which after some manipulations, we will obtain:

$$\begin{aligned}&\frac{\partial G_{\varsigma_2(t)}(v,t)}{\partial t}+\left[\lambda_2\left(1-F(v)\right)+\mu\left(1-\sum_{j=1}^{K}\frac{1}{v^j}\right)\right]G_{\varsigma_2(t)}(v,t)\\&=\mu\sum_{j=1}^{K}P_j(t)-\mu\sum_{j=1}^{K}\left[\frac{P_0(t)}{v^j}+\frac{P_1(t)}{v^{j-1}}+\frac{P_2(t)}{v^{j-2}}+\cdots+\frac{P_{j-1}(t)}{v}+P_j(t)\right].\end{aligned} \tag{12.10}$$

Equation (2.10) is an ordinary linear differential equation with respect to t, whose solution is (12.4) and hence, the proof is completed.

12.1. Moments of number of jobs in S2

As before, having the PGF of S2 given by Theorem 12.1, we find the first two moments as:

$$\begin{aligned}E\left(\zeta_2(t)\right)=\left.\frac{\partial G_{\zeta_2}(v,t)}{\partial v}\right|_{v=1}&=e^{-r(1)t}\left\{-r'(1)t\left[1+\int_0^t e^{r(1)u}D(1)du\right]\right.\\&\left.\quad+\int_0^t e^{r(1)u}\left[r'(1)uD(1)+D'(1)\right]du\right\},\end{aligned} \tag{12.11}$$

and

$$\left.\frac{\partial^2 G_{\zeta_2}(v,t)}{\partial v^2}\right|_{z=1} = e^{-r(1)t}\left\{\left[-r'(1)t\right]^2 - r''(1)t\right\}\left[1+\int_0^t D(1)du\right]$$
$$-2e^{-r(1)t}r'(1)t\int_0^t e^{r(1)u}\left[r'(1)uD(1)+D'(1)\right]du \quad (12.12)$$
$$+e^{-r(1)t}\int_0^t e^{r(1)u}\left[\begin{array}{l}\left(r'(1)u\right)^2 D(1)+r''(1)uD(1)\\ \qquad +2\left(r'(1)u\right)^2 D(1)+D''(1)\end{array}\right]du,$$

where:

$$r(v)=\lambda_2(1-F(v))+\mu-\mu\left(\frac{1}{v}+\frac{1}{v^2}+\frac{1}{v^3}\cdots+\frac{1}{v^K}\right). \quad (12.13)$$

$$r(1)=\mu(1-K). \quad (12.14)$$

$$r'(v)=-\lambda_2 F'(v)-\mu\left(-\frac{1}{v^2}-\frac{2}{v^3}-\cdots-\frac{K}{v^{K+1}}\right). \quad (12.15)$$

$$r'(1)=-\lambda_2\left(\pi_1+2\pi_2+3\pi_3+\cdots+j\pi_j+\cdots\right)+\mu\frac{K(K+1)}{2}. \quad (12.16)$$

$$r''(v)=-\lambda_2 A''(v)-\mu\left(\frac{2}{v^3}+\frac{2\cdot 3}{v^4}+\cdots+\frac{K(K+1)}{2}\right). \quad (12.17)$$

$$r''(1)=-\lambda_2\left(2\pi_2+3\cdot 2\pi_3+4\cdot 3\pi_4+\cdots+j(j-1)\pi_j+\cdots\right)$$
$$-\mu\left(2+3\cdot 2+4\cdot 3+\cdots+j(j-1)\pi_j+\cdots+K(K+1)\right). \quad (12.18)$$

$$D(v)=\mu\left\{\sum_{j=0}^{K}P_j(u)-\sum_{j=1}^{K}\left[\frac{P_0(u)}{v^j}+\frac{P_1(u)}{v^{j-1}}+\cdots+\frac{P_{j-2}(u)}{v^2}+\frac{P_{j-1}(u)}{v}+P_j(u)\right]\right\}. \quad (12.19)$$

$$D(1)=\mu\left\{\sum_{j=0}^{K}P_j(u)-\sum_{j=1}^{K}\left[P_0(u)+P_1(u)+\cdots+P_{j-2}(u)+P_{j-1}(u)+P_j(u)\right]\right\}. \quad (12.20)$$

$$D'(v)=-\mu\sum_{j=1}^{K}\left[\frac{-j}{v^{j+1}}P_0(u)+\frac{-j+1}{v^j}P_1(u)+\frac{-j+2}{v^{j-1}}P_2(u)+\cdots+\frac{-2}{v^3}P_{j-2}(u)+\frac{-1}{v^2}P_{j-1}(u)\right]. \quad (12.21)$$

$$D'(1)=\mu-\sum_{j=1}^{K}\left[jP_0(u)+(j-1)P_1(u)+(j-2)P_2(u)+\cdots+2P_{j-2}(u)+P_{j-1}(u)\right]. \quad (12.22)$$

$$D''(v)=\mu\sum_{j=1}^{K}\left[\frac{-j-1}{v^{j+2}}P_0(u)+\frac{(j-1)(-j)}{v^{j+1}}P_1(u)+\frac{(j-2)(-j+1)}{v^j}P_2(u)+\cdots+\frac{-2}{v^3}P_{j-1}(u)\right]. \quad (12.23)$$

$$D''(1) = \mu - \sum_{j=1}^{K} \left[jP_0(u) + (j-1)P_1(u) + (j-2)P_2(u) + \cdots + 2P_{j-2}(u) + P_{j-1}(u) \right]. \tag{12.24}$$

The variance of $\varsigma_2(t)$ can be obtained by

$$Var\left(\zeta_2(t)\right) = \frac{\partial^2}{\partial z^2} G_{\varsigma_2(t)}(z,t)\bigg|_{z=1} + \frac{\partial}{\partial z} G_{\varsigma_2(t)}(z,t)\bigg|_{z=1} - \left[\frac{\partial}{\partial z} G_{\varsigma_2(t)}(z,t)\bigg|_{z=1} \right]^2. \tag{12.25}$$

13. Special Case, *M/M/1, j =1*

Let $P_n(t) \equiv P\left\{\zeta_2(t) = n\right\}$ denote the probability of *n* jobs in S2 at time *t* and B2 being empty at $t = 0$. This special case is a well-known system; whose system of differential difference equations is:

$$P_0^{'}(t) = -\lambda_2 P_0(t) + \mu P_1(t), \ \ t \geq 0, \tag{13.1}$$

$$P_n^{'}(t) = -\left(\lambda_2 + \mu\right) P_n(t) + \lambda_2 \pi_1 P_{n-1}(t) + \mu P_{n+1}(t), \quad n \geq 1, \ \ t \geq 0. \tag{13.2}$$

We solve the system (13.1) and (13.2) similar to the general system (12.2) and (12.3). Thus, we will have:

$$G_{\zeta_2}(v,t) = e^{-\left[\lambda_2(1-v)+\mu\left(1-\frac{1}{v}\right)\right]t} \left\{ 1 + \mu\left(1 - \frac{1}{v}\right) \int_0^t e^{\left[\lambda_2(1-v)+\mu\left(1-\frac{1}{v}\right)\right]u} P_0(u)du \right\}. \tag{13.3}$$

Note that the system (13.1) and (13.2) is similar to the one of Parthasarathy [36]. Based on his method, he finds the PGF (according to his notations and our parameters) as:

$$H_{\zeta_2}(s,t) = H(s,0)e^{\left(\lambda_2 s + \frac{\mu}{s}\right)t} - \mu \int_0^t q_1(y) e^{\left(\lambda_2 s + \frac{\mu}{s}\right)t} (t-y)dy, \tag{13.4}$$

where

$$H(s,t) = \sum_{n=-\infty}^{\infty} q_n(t) s^n, \tag{13.5}$$

$$H(s,0) = s^l \left[\mu(1-\delta_{0l}) - \lambda_2 s\right], \tag{13.6}$$

and

$$q_n(t) = \begin{cases} e^{(\lambda_2+\mu)t}\left[\mu P_n(t) - \lambda_2 \pi_1 P_{n-1}(t)\right], & n = 1, 2, \cdots, \\ 0, & n = 0, -1, -2, \cdots. \end{cases} \tag{13.7}$$

Accordingly, we obtain the probability of the number, *n*, of jobs in S2 for this special condition and the initial value as:

$$P_n(t) = \frac{1}{\mu} e^{-(\lambda_2+\mu)t} \sum_{j=1}^{n} q_j(t) \left(\frac{\lambda_2}{\mu}\right)^{n-i} + \left(\frac{\lambda_2}{\mu}\right)^{n} P_0(t), \tag{13.8}$$

where

$$P_0(t) = 1 + \int_0^t q_1(u) e^{-(\lambda_2+\mu)u} du, \tag{13.9}$$

$$q_n(t) = \lambda_2 \left(\sqrt{\frac{\lambda_2}{\mu}}\right)^{n-1} \left[I_{n+1}(2\sqrt{\lambda_2\,\mu}\,t) - I_{n-1}(2\sqrt{\lambda_2\,\mu t}) \right], \tag{13.10}$$

$$e^{\left(\lambda_2 s + \frac{\mu}{s}\right)t} = \sum_{n=-\infty}^{\infty} \left(\sqrt{\frac{\lambda_2}{\mu}}\right) I_n\left(2\sqrt{\lambda_2 \mu}\, t\right), \tag{13.11}$$

and $I_n(t)$ is a modified Bessel function of the first kind, with the fact that $I_{-n} = I_n$, $n = 1, 2, 3, \cdots$.

13.1. Moments of the special case

Now, from (13.3), the mean number of jobs in S2 will be:

$$\frac{\partial}{\partial z} G_{\zeta_2}(v,t)\Big|_{z=1} = e^{-\lambda_2(1-\pi_1)\,t} \left[\lambda_2 \pi_1 - \mu \int_0^t e^{\lambda_2(1-\pi_1)\,t} \left(1 - \frac{1}{v}\right) P_0(u) du \right], \tag{13.12}$$

or

$$E\left(\varsigma_2\right) = (\lambda_2 - \mu)t + \mu \int_0^t P_0(u) du. \tag{13.13}$$

From (13.12), the second partial can be found as:

$$\begin{aligned} \frac{\partial^2}{\partial v^2} G_{\varsigma_2}(v,t)\Big|_{z=1} &= e^{-\lambda_2(1-\pi_1)\,t} \Big\{ \left[(\lambda_2 \pi_1 - \mu)^2 + 2\mu t \right] \\ &+ \left[2\mu(\lambda_2 \pi_1 - \mu - 1) \int_0^t e^{\lambda_2(1-\pi_1)\,t} P_0(u) du \right] \Big\}. \end{aligned} \tag{13.14}$$

From (13.13) and (13.14) the variance can be obtained.

14. The Entire System

So far, we have found PGF of the number of jobs in each stage S1 and S2. We have already defined the random variables $\zeta_1(t)$ and $\zeta_2(t)$ representing the number of jobs in S1 and S2, respectively. We have also defined $G_{\zeta_{1(i)}}(w,t)$, $G_{\zeta_{1(ii)}}(z,t)$, and $G_{\zeta_2}(w,z,y)$, as the PGF of S1(i), S1(ii) and S2, respectively. It is well-known that if $X_1, \cdots X_n$ are independent discrete random variables taking on non-negative integer values, with corresponding probability generating functions $G_{X_1}(s),\cdots,G_{X_n}(s)$, then $G_{X_1+\cdots+X_n}(s)=G_{X_1}(s)\cdots,G_{X_n}(s)$. In other words, the PGF of an independent sum is the product of the PGFs. Thus, we should be able to find the PGF of the sum of $\zeta_1(t)$ and $\zeta_2(t)$, two independent variables, which gives the PGF of the total number of jobs in the system. For a special case discussed in Sections 8 and 14, the mean of number of jobs in the entire system we can add (6.3) and (13.7) at any particular time *t*.

15. Conclusion

In this paper, a two-stage tandem time-dependent queue was considered. At Stage 1, two separate scenarios were studied. In case S1(i), the breakdown did not affect the arrival of jobs, hyperexponential distribution was used to set up the system of differential difference equations and linear first order differential equation was used to obtain the probability generating function. In the real-world, this type of scenario occurs when a computer is used and it fails at times, or a persona becomes absent from the job with variety of reasons. In case S1(ii) of Stage 1, while a service is performing, either the server becomes disabled, or the machine experiences a breakdown and fails to continue to work, we also obtained the probability generating function for this case. Hence, for S1, the PGF for the distribution and the moments of number of jobs in S1 have been found. The stationary probability generating function for this part has also been given. Further, the output of S1 was discussion that resulted the input of S2. Then, the PGF of S2 was calculated. Convolution of PGF of distributions of the number in the entire system was argued and mentioned how it can be obtained. What would be needed for future work are the waiting time, busy period and variation of the second stage. Due to the length of the paper, they were not address in this paper.

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